Journal of Mathematical Research with Applications
Mar., 2020, Vol. 40, No. 2, pp. 196–208
DOI:10.3770/j.issn:2095-2651.2020.02.008
Http://jmre.dlut.edu.cn

Application of the Alternating Direction Method for the Structure-Preserving Finite Element Model Updating Problem

Baocan ZHANG
Chengyi University College, Jimei University, Fujian 361021*, P. R. China*

Abstract This paper shows that the alternating direction method can be used to solve the structured inverse quadratic eigenvalue problem with symmetry, positive semi-definiteness and sparsity requirements. The results of numerical examples show that the proposed method works well.

Keywords finite element model updating; alternating direction method; convex programming; damped system

MR(2010) Subject Classification 65F15; 15A22; 65H17

1. Introduction

It is well-known that vibrating structures, such as bridges, highways, buildings, etc., can be mathematically modeled by a system of differential equations of the form

$$M\ddot{\mathbf{v}}(t)+C\dot{\mathbf{v}}(t)+K\mathbf{v}(t)=0 \tag{1.1}$$

where M,C and K are $n\times n$ matrices, and $\ddot{\mathbf{v}}(t)$ and $\dot{\mathbf{v}}(t)$ denote the first and second derivatives of the time-dependent vector $\mathbf{v}(t)$, respectively. Eq. (1.1) is usually obtained by discretization of a distributed parameter system with finite element techniques, and therefore, known as the finite element model. It is well known that if

$$\mathbf{v}(t)=\mathbf{x}e^{\lambda t}$$

represents a fundamental solution to (1.1), then the scalar λ and the vector $\mathbf{x}$ must solve the quadratic eigenvalue problem (QEP)

$$(\lambda^2M+\lambda C+K)\mathbf{x}=0. \tag{1.2}$$

The scalar λ and $\mathbf{x}$ are called, respectively, the eigenvalue and the eigenvector corresponding to λ. A good survey of many applications, mathematical properties, and a variety of numerical techniques for the quadratic eigenvalue problem (QEP) can be found in the treatise by Tisseur and Meerbergen [1]. For convenience, we define the λ-matrix

$$Q(\lambda)=\lambda^2M+\lambda C+K, \tag{1.3}$$

Received April 6, 2019; Accepted May 26, 2019
Supported by Youth Teacher Education and Research Funds of Fujian (Grant No. JAT170911).
E-mail address: baocanzhang@qq.com

which has $2n$ eigenvalues and $2n$ eigenvectors when M is nonsingular. The eigenvalues of $Q(\lambda)$ are related to the natural frequencies of the homogeneous system and the eigenvectors are the mode shapes of the vibration of the system [2]. The matrices M, C and K are called the mass, damping and stiffness matrices, respectively. And they often possess exploitable structure properties such as symmetry, definiteness, sparsity/bandedness, etc.

The model updating problems (MUP) concern updating of a finite element model in such a way that a set of "unwanted eigenvalues and eigenvectors" from the original model is replaced by the suitably given or measured ones and some important physical properties of the original model are preserved, for example, the symmetry, the definiteness and the sparsity. Various model updating problems with (M, C, K) inheriting different structures have been considered in the literature. See Baruch [3], Wei [4], Xie [5] for the undamped case when $C = 0$. For an account of the earlier methods, see the book by Friswell and Mottershead [6], an integral introduction of the basic theory of finite element model updating is given. For damped structured systems, Carvalho [7] for the symmetric eigenvalue embedding problem, Friswell [8], Kuo [9], Lancaster [10] for the most commonly discussed case when M is positively definite and (C, K) are symmetric, Chu [11] and Bai [12] for the case (M, K) are positive definite and semi-positive definite, respectively. Xiao [13] considered the model updating problem with damped gyroscopic structure with M, K being positive definite. In [14], Joali showed that the alternating projection method can be used to solve the matrix model updating problem which preserves the symmetry of the original model, but they did not consider the preservation of the other important physical properties, such as the definiteness and sparsity of the original model. Recently, Zhao [15, 16] considered the inverse eigenvalue problems for the quadratic palindromic systems with partially prescribed eigenstructure. All the methods mentioned above do not take the structural connectivity into consideration, that is, the updated mass matrix, damping matrix and stiffness matrix do not preserve the sparsity or the zero/nonzero patterns of the original analytical model. To overcome this shortcoming, Kabe [17] developed an algorithm for undamped system which preserves the connectivity of the structural mode. Recently, Bai [18] proposed a smoothing Newton-type algorithm for the case that (M, C, K) are all symmetric positive definite matrices with special structures. However, Kabe and Bai's methods cannot preserve the symmetry, positive semi-definiteness and sparsity simultaneously for the damping system.

In this paper, we update the mass matrix, damping matrix and stiffness matrix with requirements of satisfaction of the characteristic equation, symmetry, positive semi-definiteness and sparsity simultaneously. Such a problem is formulated in the following form:

$$\begin{array}{ll} \min & \frac{1}{2}\|M - M_a\|_F^2 + \frac{1}{2}\|C - C_a\|_F^2 + \frac{1}{2}\|K - K_a\|_F^2 \\ \text{s.t.} & M\widetilde{X}\widetilde{\Lambda}^2 + C\widetilde{X}\widetilde{\Lambda} + K\widetilde{X} = 0, \\ & M^T = M, C^T = C, K^T = K, \\ & M \geq 0, K \geq 0, \\ & \text{sparse}(M) = \text{sparse}(M_a), \\ & \text{sparse}(C) = \text{sparse}(C_a), \\ & \text{sparse}(K) = \text{sparse}(K_a), \end{array} \tag{1.4}$$

where M, K, C are the unknown variables, and M_a, K_a, C_a are mass, stiffness and damping matrices of analytical model respectively, and $\widetilde{\Lambda} = \mathrm{diag}\{\lambda_1, \dots, \lambda_p\} \in C^{p\times p}$, $\widetilde{X} = [x_1, \dots, x_p] \in C^{n\times p} (p \ll n)$ and both $\widetilde{\Lambda}$ and $\widetilde{X}$ are closed under complex conjugation in the sense that $\lambda_{2j} = \overline{\lambda}_{2j-1} \in C$, $x_{2j} = \overline{x}_{2j-1} \in C^n$ for $j = 1, \dots, l$ and $\lambda_k \in R, x_k \in R^n$ for $k = 2l+1, \dots, p$. Sparse(M) =sparse(M_a), sparse(C) =sparse(C_a) and sparse(K) =sparse(K_a) denote the sparsity requirements on the mass matrix M, the damping matrix C and the stiffness matrix K to be updated. In the finite element model updating literature, $\widetilde{\Lambda}$ and $\widetilde{X}$ are referred to as measured eigenvalue and eigenvector matrices, because in finite element model updating setting, as set of experimentally measured data is needed to be incorporated into an updated finite element model. Simply, we suppose that $\widetilde{X}$ is of full column rank.

Throughout this paper, the following notations will be used. For $A \in R^{m\times n}$, A^T denotes the transpose of A, $\mathrm{tr}(A)$ stands for the trace of A, $\mathrm{rank}(A)$ denotes the rank of A. For $A \in R^{m\times n}$ and $B \in R^{m\times n}$, $\langle A, B\rangle = \mathrm{tr}(A^T B)$ and $A \circ B$ represents the Hadamard product of A and B. $SR^{n\times n}$ is the set of all $n \times n$ symmetric matrices and $SR^{n\times n}_{r,+}$ is the set of all $n \times n$ symmetric matrices with its $r \times r$ leading submatrix positive semi-definite. I_n is the $n \times n$ identity matrix and $\|\cdot\|_F$ is the Frobenius norm. We write $A \geq 0$ if A is a real symmetric positive semi-definite matrix.

This paper is organized as follows. In Section 2, we first rewrite the problem (1.4) equivalently, and then an accelerated alternating direction method (ADM) for problem (1.4) is presented. We implement our method in practice and report the numerical results in Section 3. Finally, we conclude this paper in Section 4.

2. Application of ADM to problem (1.4)

To present the ADM method for problem (1.4), we redescribe the sparsity requirements on M_a, K_a and C_a. Define the index set

$$I_m = \{(i,j) | m_{ij} = 0, M_a = (m_{ij}) \in R^{n\times n}\},$$

$$I_k = \{(i,j) | k_{ij} = 0, K_a = (k_{ij}) \in R^{n\times n}\},$$

$$I_c = \{(i,j) | c_{ij} = 0, C_a = (c_{ij}) \in R^{n\times n}\},$$

and auxiliary matrices $T_1 = (t_{ij}) \in R^{n\times n}$, $T_2 = (t'_{ij}) \in R^{n\times n}$ and $T_3 = (t''_{ij}) \in R^{n\times n}$ depending on the sparsity requirements on M_a, K_a and C_a, respectively, as follows:

$$t_{ij} = \begin{cases} 0, & (i,j) \notin I_m, \\ 1, & (i,j) \in I_m. \end{cases} \tag{2.1}$$

$$t'_{ij} = \begin{cases} 0, & (i,j) \notin I_k, \\ 1, & (i,j) \in I_k. \end{cases} \tag{2.2}$$

$$t''_{ij} = \begin{cases} 0, & (i,j) \notin I_c, \\ 1, & (i,j) \in I_c. \end{cases} \tag{2.3}$$

It is obvious that T_1, T_2 and T_3 depend on M_a, C_a and K_a, respectively. Then the sparsity requirements on M, K and C have an equivalent expression

$$\begin{aligned}\text{sparse}(M) = \text{sparse}(M_a) \Leftrightarrow M \circ T_1 = 0,\\ \text{sparse}(K) = \text{sparse}(K_a) \Leftrightarrow K \circ T_2 = 0,\\ \text{sparse}(C) = \text{sparse}(C_a) \Leftrightarrow C \circ T_3 = 0,\end{aligned}$$

where $\circ$ stands for the Hadamard product.

Define a matrix T_p as

$$T_p = \text{diag}\left\{\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\ 1 & i\end{bmatrix}, \dots, \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\ 1 & i\end{bmatrix}, I_{p-2l}\right\} \in C^{p\times p},$$

where $i = \sqrt{-1}$. It is easy to verify that T_p is a unitary matrix, i.e., $\overline{T}^T T = I_p$. By this transformation matrix, we have

$$\Lambda = \overline{T}_p^T \widetilde{\Lambda} T_p = \text{diag}\left\{\begin{bmatrix}\xi_1 & \eta_1\\ -\eta_1 & \xi_1\end{bmatrix}, \dots, \begin{bmatrix}\xi_l & \eta_l\\ -\eta_l & \xi_l\end{bmatrix}, \lambda_{2l+1}, \dots, \lambda_p\right\} \in R^{p\times p}, \tag{2.4}$$

$$X = \widetilde{X} T_p = [\sqrt{2}y_1, \sqrt{2}z_1, \dots, \sqrt{2}y_{2l-1}, \sqrt{2}z_{2l-1}, x_{2l+1}, \dots, x_p] \in R^{n\times p}. \tag{2.5}$$

By (2.4) and (2.5), the dynamic equation of (1.4) can be written equivalently as

$$MX\Lambda^2 + CX\Lambda + KX = 0. \tag{2.6}$$

Let the block matrices $Y \in R^{3n\times 3n}$, $Y_a \in R^{3n\times 3n}$, $W \in R^{3n\times p}$ and $E \in R^{3n\times n}$ be defined as

$$Y = \begin{pmatrix} M & & \\ & K & \\ & & C\end{pmatrix}, \quad Y_a = \begin{pmatrix} M_a & & \\ & K_a & \\ & & C_a\end{pmatrix}, \quad E = \begin{pmatrix} I_n\\ I_n\\ I_n\end{pmatrix}, \quad W = \begin{pmatrix} X\Lambda^2\\ X\\ X\Lambda\end{pmatrix}. \tag{2.7}$$

Clearly,

$$MX\Lambda^2 + CX\Lambda + KX = (M, K, C)W = E^T Y W.$$

Let

$$T = \begin{pmatrix} T_1 & T_{12} & T_{13}\\ T_{12} & T_2 & T_{23}\\ T_{13} & T_{23} & T_3\end{pmatrix}, \tag{2.8}$$

where T_{12}, T_{13}, T_{23} are the $n\times n$ matrices with every (i,j)-entry being 1 for all $i, j = 1, \dots, n$, and T_1, T_2 and T_3 are defined by (2.1)–(2.3), respectively. Denote the feasible region of problem (1.4) by D. Let

$$\begin{aligned}S_1 &= \{Y \in R^{3n\times 3n} | E^T Y W = 0, Y \circ T = 0\},\\ S_2 &= \{Y \in R^{3n\times 3n} | Y \in SR^{3n\times 3n}_{2n,+}\}.\end{aligned}$$

Clearly, $D = S_1 \cap S_2$. Then problem (1.4) can be equivalently rewritten as

$$\begin{aligned}\min \quad & \tfrac{1}{2}\|Y - Y_a\|_F^2 + \tfrac{1}{2}\|Z - Y_a\|_F^2\\ \text{s.t.} \quad & Y - Z = 0,\\ & Y \in S_1, Z \in S_2.\end{aligned} \tag{2.9}$$

Note that the feasible region D is a closed convex set, and it follows from the best approximation theory that problem (2.9) has a unique solution if D is nonempty. Throughout this paper, we always assume that D is nonempty.

To show how ADM can be applied to (2.9), let the augmented Lagrangian function of (2.9) be given by

$$L_\beta(Y,Z,\Delta)=\frac{1}{2}\|Y-Y_a\|_F^2+\frac{1}{2}\|Z-Y_a\|_F^2-<\Delta,Y-Z>+\frac{\beta}{2}\|Y-Z\|_F^2, \tag{2.10}$$

which is defined on $\Omega=SR_{2n,+}^{3n\times 3n}\times R^{3n\times 3n}\times R^{3n\times 3n}$, where $\beta>0$ is a penalty parameter on the linear constraint and $\Delta\in R^{3n\times 3n}$ is the Lagrange multiplier. Thus, we can write the iterative scheme of ADM for (2.9) as

$$\begin{cases} Y^{k+1}=\mathrm{argmin}_{Y\in S_1}\{L_\beta(Y,Z^k,\Delta^k)\},\\ Z^{k+1}=\mathrm{argmin}_{Z\in S_2}\{L_\beta(Y^{k+1},Z,\Delta^k)\},\\ \Delta^{k+1}=\Delta^k-\beta(Y^{k+1}-rZ^{k+1}).\end{cases} \tag{2.11}$$

At each iteration, the scheme (2.11) requires to handle two subproblems which are much easier than the original model (1.4): one is a simple linear least-squares problem with symmetric constraint while the other is a standard approximate problem with sparsity constraint. The main computation effort of (2.11) is to solve Y^{k+1} and Z^{k+1}, which can be simplified as

$$Y^{k+1}=\mathrm{argmin}_{Y\in S_1}\{\frac{1}{2}\|Y-\frac{1}{\beta+1}(Y_a+\Delta^k+\beta Z^k)\|_F^2\}, \tag{2.12}$$

$$Z^{k+1}=\mathrm{argmin}_{Z\in S_2}\{\frac{1}{2}\|Z-\frac{1}{\beta+1}(Y_a-\Delta^k+\beta Y^{k+1})\|_F^2\}, \tag{2.13}$$

respectively. In the following part of this section, we discuss how to solve the subproblems (2.12) and (2.13), respectively.

2.1. Solving the subproblem (2.12)

The problem (2.12) is equivalent to

$$\begin{array}{ll}\min & \frac{1}{2}\|Y-\widetilde{Y}\|_F^2\\ \text{s.t} & E^TYW=0, Y\circ T=0,\end{array} \tag{2.14}$$

where $\widetilde{Y}=\frac{1}{\beta+1}(Y_a+\Delta^k+\beta Z^k)$. Partition $\widetilde{Y}$ as $\widetilde{Y}=(\widetilde{Y}_{ij})$, where $\widetilde{Y}_{11},\widetilde{Y}_{22},\widetilde{Y}_{33}\in R^{n\times n}$. Denote the i-th row of M, K,C, $\widetilde{Y}_{11}$, $\widetilde{Y}_{22}$ and $\widetilde{Y}_{33}$ by $M^{(i)}$, $K^{(i)}$, $C^{(i)}$, $\widetilde{Y}_{11}^i$, $\widetilde{Y}_{22}^{(i)}$ and $\widetilde{Y}_{33}^{(i)}$, respectively. By the definition of Y_a and sparse(Y) =sparse(Y_a), we have

$$\begin{aligned}\|Y-\widetilde{Y}\|_F^2=&\|M-\widetilde{Y}_{11}\|_F^2+\|K-\widetilde{Y}_{22}\|_F^2+\|C-\widetilde{Y}_{33}\|_F^2+\\ &\|\widetilde{Y}_{12}\|_F^2+\|\widetilde{Y}_{21}\|_F^2+\|\widetilde{Y}_{13}\|_F^2+\|\widetilde{Y}_{31}\|_F^2+\|\widetilde{Y}_{23}\|_F^2+\|\widetilde{Y}_{32}\|_F^2.\end{aligned}$$

Hence $\|Y-\widetilde{Y}\|_F^2$ is minimized if and only if $\|(M,K,C)-(\widetilde{Y}_{11},\widetilde{Y}_{22},\widetilde{Y}_{33})\|_F^2$ is minimized. Since $E^TYW=(M,K,C)W$, problem (2.14) can be separated into n independent subproblems of the same structure

$$\begin{array}{ll}\min & \Sigma_{i=1}^n\frac{1}{2}\|(M^{(i)},K^{(i)},C^{(i)})-(\widetilde{Y}_{11}^{(i)},\widetilde{Y}_{22}^{(i)},\widetilde{Y}_{33}^{(i)})\|_F^2\\ \text{s.t} & (M^{(i)},K^{(i)},C^{(i)})W=0, i=1,2,\ldots,n.\end{array}$$

Now, we focus on the following i-th subproblem

$$\begin{array}{ll}\min & \frac{1}{2}\|(M^{(i)},K^{(i)},C^{(i)})-(\widetilde{Y}_{11}^{(i)},\widetilde{Y}_{22}^{(i)},\widetilde{Y}_{33}^{(i)})\|_F^2 \\ \text{s.t} & (M^{(i)},K^{(i)},C^{(i)})W=0,\end{array} \tag{2.15}$$

Without loss of generality, we assume that there are n_i nonzero elements in the i-th row of (M_a, K_a, C_a). There exist three permutation matrices P_i', Q_i', $S_i' \in R^{n\times n}$, which are determined by the sparsity of M_a, K_a and C_a, respectively, such that

$$M^{(i)} = (M_1^{(i)}, 0)P_i', K^{(i)} = (K_1^{(i)}, 0)Q_i', C^{(i)} = (C_1^{(i)}, 0)S_i',$$

which imply that

$$(M^{(i)}, K^{(i)}, C^{(i)}) = ((M_1^{(i)}, 0), (K_1^{(i)}, 0), (C_1^{(i)}, 0))\text{diag}\{P_i', Q_1', S_1'\}.$$

It is easy to see that there exists a permutation matrix $Q_i \in R^{3n\times 3n}$ such that

$$(M^{(i)}, 0, K_1^{(i)}, 0, C_1^{(i)}, 0) = (M_1^{(i)}, K_1^{(i)}, C_1^{(i)}, 0, 0, 0)Q_i.$$

Let $P_i = Q_i\text{diag}\{P_i', Q_1', S_1'\}$, $R^{(i)} = (M^{(i)}, K^{(i)}, C^{(i)}) \in R^{1\times 3n}$ and $R_1^{(i)} = (M_1^{(i)}, K_1^{(i)}, C_1^{(i)}) \in R^{1\times n_i}$. Then

$$R^{(i)} = (R_1^{(i)}, 0)P_i. \tag{2.16}$$

Partition

$$P_iW = \begin{pmatrix} W^{(i_1)} \\ W^{(i_2)} \end{pmatrix}, \tag{2.17}$$

where $W^{(i_1)} \in R^{n_i\times p}$, which imply that

$$0 = R^{(i)}W = (R_1^{(i)}, 0)P_iW = R_1^{(i)}W^{(i_1)}, \tag{2.18}$$

Assume that $\text{rank}(W^{(i_1)}) = r_i < n_i$ and the QR decomposition of $W^{(i_1)}$ is

$$W^{(i_1)} = U_i \begin{pmatrix} \Sigma_i \\ 0 \end{pmatrix}, \tag{2.19}$$

where $\Sigma_i \in R^{r_i\times p}$ with $\text{rank}(\Sigma_i) = r_i$, $U_i \in R^{n_i\times n_i}$ is an orthogonal matrix. Substituting (2.19) into (2.18), we obtain that

$$R_1^{(i)}W^{(i_1)} = R_1^{(i)}U_i \begin{pmatrix} \Sigma_i \\ 0 \end{pmatrix} = 0.$$

Partition $R_1^{(i)}U_i$ according to (2.19) as $R_1^{(i)}U_i = ((R_1^{(i)}U_i)_1, (R_1^{(i)}U_i)_2)$, where $(R_1^{(i)}U_i)_1 \in R^{1\times r_i}$. It follows that $R_1^{(i)}W^{(i_1)} = 0$ if and only if $(R_1^{(i)}U_i)_1 = 0$, which imply that the solution of (2.18) is

$$R_1^{(i)} = (0, (\widetilde{R}_1^{(i)})_2)U_i^T,$$

where $(\widetilde{R}_1^{(i)})_2 \in R^{1\times(n_i-r_i)}$ is arbitrary.

Now, we consider the objective function of problem (2.15). Denote

$$(\widetilde{Y}_{11}^{(i)}, \widetilde{Y}_{22}^{(i)}, \widetilde{Y}_{33})P_i^T = (\tilde{y}_1^{(i)}, \tilde{y}_2^{(i)}), \tag{2.20}$$

where $\tilde{y}_1^{(i)} \in R^{1\times n_i}$ and partition $\tilde{y}_1^{(i)}U_i$ according to $R_1^{(i)}$ as

$$\tilde{y}_1^{(i)}U_i = ((\tilde{y}_1^{(i)})_1, (\tilde{y}_1^{(i)})_2). \tag{2.21}$$

Thus, we have

$$\begin{aligned}
&\|(M^{(i)}, K^{(i)}, C^{(i)}) - (\widetilde{Y}_{11}^{(i)}, \widetilde{Y}_{22}^{(i)}, \widetilde{Y}_{33}^{(i)})\|_F^2 \\
&= \|(R_1^{(i)}, 0)P_i - (\widetilde{Y}_{11}^{(i)}, \widetilde{Y}_{22}^{(i)}, \widetilde{Y}_{33}^{(i)})\|_F^2 \\
&= \|(R_1^{(i)}, 0) - (\widetilde{Y}_{11}^{(i)}, \widetilde{Y}_{22}^{(i)}, \widetilde{Y}_{33}^{(i)})P_i^T\|_F^2 \\
&= \|(R_1^{(i)}, 0) - (\tilde{y}_1^{(i)}, \tilde{y}_2^{(i)})\|_F^2 \\
&= \|R_1^{(i)} - \tilde{y}_1^{(i)}\|_F^2 + \|\tilde{y}_2^{(i)}\|_F^2 \\
&= \|(0, (\widetilde{R}_1^{(i)})_2)U_i^T - \tilde{y}_1^{(i)}\|_F^2 + \|\tilde{y}_2^{(i)}\|_F^2 \\
&= \|(0, (\widetilde{R}_1^{(i)})_2) - ((\tilde{y}_1^{(i)})_1, (\tilde{y}_1^{(i)})_2)\|_F^2 + \|\tilde{y}_2^{(i)}\|_F^2 \\
&= \|(\tilde{y}_1^{(i)})_1\|_F^2 + \|(\widetilde{R}_1^{(i)})_2 - (\tilde{y}_1^{(i)})_2\|_F^2 + \|\tilde{y}_2^{(i)}\|_F^2.
\end{aligned}$$

It is easy to verify that $\|(M^{(i)}, K^{(i)}, C^{(i)}) - (\widetilde{Y}_{11}^{(i)}, \widetilde{Y}_{22}^{(i)}, \widetilde{Y}_{33})\|_F^2$ is minimized if and only if $\|(\widetilde{R}_1^{(i)})_2 - (\tilde{y}_1^{(i)})_2)\|_F^2$ is minimized, which imply that

$$(\widetilde{R}_1^{(i)})_2 = (\tilde{y}_1^{(i)})_2.$$

Thus we have the following result.

Theorem 2.1 *Let $M_a, C_a, K_a \in SR^{n\times n}, \widetilde{\Lambda} \in R^{p\times p}$, $M_a \geq 0, K_a \geq 0$ and $\widetilde{X} \in R^{n\times p}$ $(p \ll n)$. P_i is the permutation matrix defined by Eq. (2.16). Denote the ith row of M, K, C, $\widetilde{Y}_{11}$, $\widetilde{Y}_{22}$ and $\widetilde{Y}_{33}$ by $M^{(i)}$, $K^{(i)}$, $C^{(i)}$, $\widetilde{Y}_{11}^{i}$, $\widetilde{Y}_{22}^{(i)}$ and $\widetilde{Y}_{33}^{(i)}$, respectively. Partition P_iW, $(\widetilde{Y}_{11}^{(i)}, \widetilde{Y}_{22}^{(i)}, \widetilde{Y}_{33}^{(i)})P_i^T$ and $\tilde{y}_1^{(i)}U_i$ as Eqs. (2.17), (2.20) and (2.21), respectively. Assume that $W^{(i_1)}$ has QR decomposition (2.19). Then the solution of problem (2.15) is given by*

$$(M^{(i)}, K^{(i)}, C^{(i)}) = ((0, (\tilde{y}_1^{(i)})_2)U_i^T, 0)P_i, \quad i = 1, 2, \ldots, n. \tag{2.22}$$

2.2. Solving the subproblem (2.13)

The problem (2.13) is equivalent to

$$\begin{array}{ll} \min & \frac{1}{2}\|Z - \widetilde{Z}\|_F^2 \\ \text{s.t.} & Z \in SR_{2n,+}^{3n\times 3n}, \end{array} \tag{2.23}$$

where $\widetilde{Z} = \frac{1}{\beta+1}(Y_a - \Delta^k + \beta Y^{k+1})$. Let $\widehat{Z} = \frac{\widetilde{Z}+\widetilde{Z}^T}{2}$. Clearly, $\widehat{Z} \in SR^{3n\times 3n}$. According to [19], the spectral decomposition of $\widehat{Z}$ is

$$\widehat{Z} = Q\Theta Q^T, \quad \Theta = \mathrm{diag}(\theta_1, \ldots, \theta_{3n}),$$

where $Q \in R^{3n\times 3n}$ is an orthogonal matrix of orthogonal eigenvectors of $\widehat{Z}$ and $\{\theta_j\}_{j=1}^{3n}$ are eigenvalues of $\widehat{Z}$. Then, using the Frobenius inner product, the problem (2.23) has the following explicit analytic formula [20]

$$Z = Q\Theta_+Q^T, \quad \Theta_+ = \mathrm{diag}(\max\{\theta_1, 0\}, \ldots, \max\{\theta_{2n}, 0\}, \theta_{2n+1}, \ldots, \theta_{3n}).$$

2.3. The Algorithm for the problem (2.9)

Now, we are ready to present the algorithm for the problem (2.9). For simplicity, the iterate generated by the ADM scheme (2.11) is now denoted by $\widetilde{Y}^k$, $\widetilde{Z}^k$ and $\widetilde{\Delta}^k$. Then we have

$$\begin{cases} \widetilde{Y}^k = \mathrm{argmin}_{Y\in S_1}\{\frac{1}{2}\|Y - \frac{1}{\beta+1}(Y_a + \Delta^k + \beta Z^k)\|_F^2\}, \\ \widetilde{Z}^k = \mathrm{argmin}_{Z\in S_2}\{\frac{1}{2}\|Z - \frac{1}{\beta+1}(Y_a - \Delta^k + \beta \widetilde{Y}^k)\|_F^2\}, \\ \widetilde{\Delta}^k = \Delta^k - \beta(\widetilde{Y}^k - \widetilde{Z}^k), \\ Y^{k+1} = \widetilde{Y}^k, \\ Z^{k+1} = \widetilde{Z}^k, \\ Y^{k+1} = \widetilde{Y}^k. \end{cases} \tag{2.24}$$

Recently, it was proposed in [21] that the acceleration technique in the proximal point algorithm (PPA) can be used to accelerate ADM. And in [12] this technique is also used for the semidefinite inverse quadratic eigenvalue problem. We thus adopt this recent technique and propose the following accelerate ADM method for the problem (2.9).

Algorithm 1 (An accelerate ADM method). Let $\beta > 0$ and $\gamma \in (0, 2)$. With the given iterate Y^k, Z^k, Δ^k, the iterate Y^{k+1}, Z^{k+1} and Δ^{k+1} are generated as follows.

Step 1. PPA step. Obtain $\widetilde{Y}^k$, $\widetilde{Z}^k$ and $\widetilde{\Delta}^k$ via

$$\begin{cases} \widetilde{Y}^k = \mathrm{argmin}_{Y\in S_1}\left\{\frac{1}{2}\|Y - \frac{1}{\beta+1}(Y_a + \Delta^k + \beta Z^k)\|_F^2\right\}, \\ \widetilde{\Delta}^k = \Delta^k - \beta(\widetilde{Y}^k - Z^k), \\ \widetilde{Z}^k = \mathrm{argmin}_{Z\in S_2}\left\{\frac{1}{2}\|Z - \frac{1}{\beta+1}(Y_a - \widetilde{\Delta}^k + \beta \widetilde{Y}^k)\|_F^2\right\}. \end{cases} \tag{2.25}$$

Step 2. Relaxation step.

$$\begin{aligned} Z^{k+1} &= Z^k - \gamma(Z^k - \widetilde{Z}^k), \\ \Delta^{k+1} &= \Delta^k - \gamma(\Delta^k - \widetilde{\Delta}^k). \end{aligned} \tag{2.26}$$

2.4. Convergence

In [21], the convergence of the accelerate ADM in vector type is presented, which can be applied to problem (2.9) directly.

Theorem 2.2 *If the solution set of problem (1.4), denoted by D^*, is nonempty, then the sequence $\{Y^k, Z^k, \Delta^k\}$ generated by the algorithm 1 converges to a solution of problem (1.4).*

3. Numerical example

In this section, three numerical examples are presented to verify the algorithm.

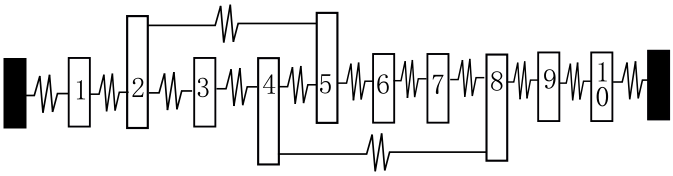

Figure 1 Mass spring system

Example 3.1 In this example we consider the model updating problem for the mass-spring system with damping which is shown in Figure 1. And the exact structured matrices (mass matrix, damping matrix and stiffness matrix) are given by $\widehat{M} = 4.0 * I_{10}$,

$$\widehat{K} = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 \end{bmatrix},$$

$$\widehat{C} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

We choose two eigenpairs $\{\lambda_j, x_j\}_{j=1}^2$ of the pencil $\widehat{Q}(\lambda) := \lambda^2 \widehat{M} + \lambda \widehat{C} + \widehat{K}$ as the measured eigendata, where

$$\lambda_{1,2} = -0.4565 \pm 1.1628i, \quad \text{and} \quad x_{1,2} = \begin{bmatrix} 0.2028 \mp 0.0559i \\ -0.3513 \pm 0.1876i \\ 0.4208 \mp 0.1343i \\ -0.3699 \pm 0.2354i \\ 0.3229 \mp 0.2260i \\ -0.1488 \pm 0.0065i \\ -0.0507 \pm 0.1521i \\ 0.1792 \mp 0.3085i \\ -0.1960 \pm 0.1534i \\ 0.1188 \mp 0.0563i \end{bmatrix}.$$

To illustrate the proposed method, we set $M_a = 1.05 * \widehat{M}, K_a = 1.2 * \widehat{K}, C_a = 1.2 * \widehat{C}$. Let $\beta = 2$, $\gamma = 1.5$ and the stopping criterion is $\|M^k X \Lambda^2 + C^k X \Lambda + K^k X\|_F < \epsilon = 10^{-11}$. By Algorithm 1, after 28 iterations and $0.317838s$ (CPU times), we get the updated matrices M, C and K as follows: $M = \mathrm{diag}\{4.4964, 4.4977, 4.5219, 4.5085, 4.5112, 4.4132, 4.4601, 4.5046, 4.5100, 4.4969\}$,

$$C = \begin{bmatrix} 2.3252 & -1.0722 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 \\ -1.0722 & 2.3050 & -1.1051 & -0.0000 & 0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000 \\ 0.0000 & -1.1051 & 2.2968 & -1.0964 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 \\ 0.0000 & -0.0000 & -1.0964 & 2.2917 & -1.1058 & 0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.0000 \\ -0.0000 & 0.0000 & 0.0000 & -1.1058 & 2.3156 & -1.0390 & 0.0000 & -0.0000 & -0.0000 & 0.0000 \\ -0.0000 & -0.0000 & -0.0000 & 0.0000 & -1.0390 & 2.3509 & -1.2191 & -0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -0.0000 & 0.0000 & 0.0000 & -1.2191 & 2.3278 & -1.1344 & 0.0000 & -0.0000 \\ -0.0000 & -0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0000 & -1.1344 & 2.3052 & -1.0855 & -0.0000 \\ -0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000 & 0.0000 & -1.0855 & 2.3252 & -1.0745 \\ 0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & -1.0745 & 2.3205 \end{bmatrix},$$

$$K = \begin{bmatrix} 4.6539 & -1.0477 & 0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 \\ -1.0477 & 4.6648 & -1.0599 & -0.0000 & -1.0583 & -0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 \\ 0.0000 & -1.0599 & 4.6540 & -1.0643 & -0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.0000 & -0.0000 \\ -0.0000 & -0.0000 & -1.0643 & 4.6657 & -1.0560 & 0.0000 & -0.0000 & -1.0640 & -0.0000 & -0.0000 \\ 0.0000 & -1.0583 & -0.0000 & -1.0560 & 4.6499 & -1.1025 & -0.0000 & -0.0000 & -0.0000 & 0.0000 \\ 0.0000 & -0.0000 & 0.0000 & 0.0000 & -1.1025 & 4.6921 & -1.2470 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & -1.2470 & 4.6755 & -1.0148 & -0.0000 & 0.0000 \\ -0.0000 & -0.0000 & 0.0000 & -1.0640 & -0.0000 & 0.0000 & -1.0148 & 4.6603 & -1.0673 & 0.0000 \\ -0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0000 & -0.0000 & -1.0673 & 4.6451 & -1.0461 \\ 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & -1.0461 & 4.6563 \end{bmatrix}.$$

The residual of the dynamic equations is $\|MX\Lambda^2 + CX\Lambda + KX\|_F = 8.1843e-12$, where $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2\}$ and $X = [x_1, x_2]$, and $\|M - M_a\|_F = 0.9286$, $\|C - C_a\|_F = 0.5305$, $\|K - K_a\|_F = 0.7801$. Clearly, the updated matrices M, C and K have the same sparsity with M_a, C_a and K_a, respectively, and the eigenvalues of K are $1.9921, 7.1709, 6.7355, 5.9087, 5.4963, 2.9026, 4.8493, 4.3147, 3.3673, 3.8801$ which imply that $K \geq 0$.

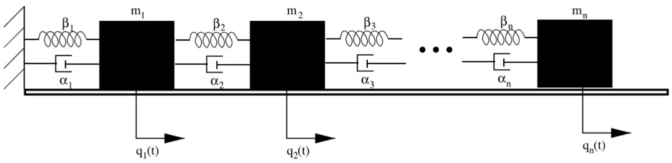

Figure 2 Serially linked mass spring system

Therefore, the measured eigenvalues and eigenvectors are embedded in the new model $(\lambda^2 M + \lambda C + K)x = 0$ which have the same structure (symmetry, positive semi-definiteness and sparsity) with the analytical model.

Example 3.2 In this example, we consider the model updating problem of the serially linked mass spring system with damping, including the spring stiffness, mass and damping value, which is shown in Figure 2. And the mass matrix, damping matrix and stiffness matrix are given by

$$C = \begin{bmatrix} \alpha_1+\alpha_2 & -\alpha_2 & & & \\ -\alpha_2 & \alpha_2+\alpha_3 & -\alpha_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -\alpha_n & \alpha_n \end{bmatrix}, \quad K = \begin{bmatrix} \beta_1+\beta_2 & -\beta_2 & & & \\ -\beta_2 & \beta_2+\beta_3 & -\beta_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -\beta_n & \beta_n \end{bmatrix},$$

and $M = \mathrm{diag}\{m_1, \ldots, m_n\}$, repectively [22].

Let $m_j = 2$, $\alpha_j = \beta_j = 1$, $j = 1, 2, \ldots, n$. Then we get the exact mass matrix $\widetilde{M}$, stiffness matrix $\widetilde{K}$ and damping matrix $\widetilde{C}$. It is easy to verify that $\widetilde{M} > 0$, $\widetilde{K} \geq 0$. We choose p eigenpairs $\{\lambda_j, x_j\}_{j=1}^{p}$ of the pencil $\widetilde{Q}(\lambda) := \lambda^2\widetilde{M} + \lambda\widetilde{C} + \widetilde{K}$ as the given measured eigendata. To illustrate the proposed method, we set

$$M_a = \widetilde{M} + \mu R_M \circ \widetilde{M}, C_a = \widetilde{C} + \mu R_C \circ \widetilde{C}, K_a = \widetilde{K} + \mu R_K \circ \widetilde{K},$$

where $\circ$ stands for Hadamard product, R_M, R_C and R_K are $n \times n$ symmetric matrices whose entries are generated pseudo-randomly and they are uniformly distributed within $[-1.0, 1.0]$, $\mu \in R$ is a perturbed parameter. The stoping criterion is

$$\|M^k X\Lambda^2 + C^k X\Lambda + K^k X\|_F / (\|M_a X\Lambda^2 + C_a X\Lambda + K_a X\|_F) < \epsilon.$$

Our numerical results are given in Tables 1 and 2, where $IT.$ and $RES.$ stand for the number of iterations and the value of $\|MX\Lambda^2+CX\Lambda+KX\|_F/(\|M_aX\Lambda^2+C_aX\Lambda+K_aX\|_F)$, respectively. The numerical results show that our proposed algorithm works well.

$p=2,\beta=10,\gamma=1.5$						
ϵ	10^{-5}		10^{-7}		10^{-9}	
n	$IT.$	$RES.$	$IT.$	$RES.$	$IT.$	$RES.$
50	6	6.3978e-06	11	5.0478e-08	18	3.1800e-10
100	6	6.3180e-06	11	4.2031e-08	16	9.9797e-10
150	6	6.3093e-06	11	4.0564e-08	16	7.9064e-10
200	6	6.3071e-06	11	4.4324e-08	15	9.7905e-10

Table 1 Numerical results of Example 3.2

$p=4,\beta=10,\gamma=1.5$						
ϵ	10^{-5}		10^{-6}		10^{-7}	
n	$IT.$	$RES.$	$IT.$	$RES.$	$IT.$	$RES.$
50	7	6.1871e-06	32	9.9335e-07	243	9.9824e-8
100	6	9.1586e-06	17	9.5300e-07	208	9.9564e-8
150	6	8.9927e-06	10	8.8774e-07	120	9.9845e-8
200	6	8.9497e-06	9	7.5133e-07	74	9.9580e-8

Table 2 Numerical results of Example 3.2

Example 3.3 To illustrate the updated matrices have the same sparsity and positive semi-definiteness of analytical matrices in original model, we only focus on the case of Example 3.2 with $n=500$, $p=4$. The stoping criterion ($\epsilon=10^{-6}$) is defined just as in Example 3.2.

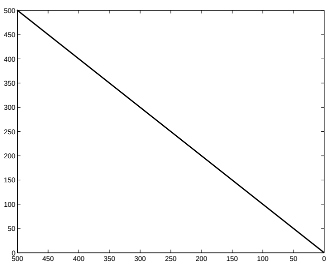

Figure 3 Zero/nonzero pattern of the updated mass matrix M

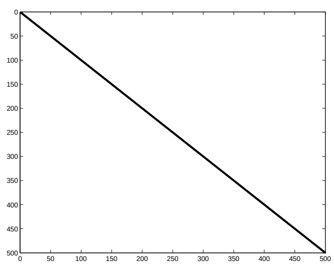

Figure 4 Zero/nonzero pattern of the updated damping matrix C

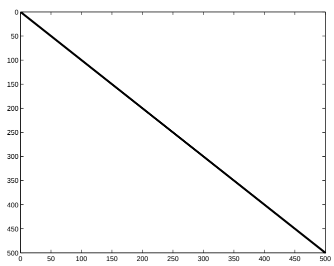

Figure 5 Zero/nonzero pattern of the updated stiffness matrix K

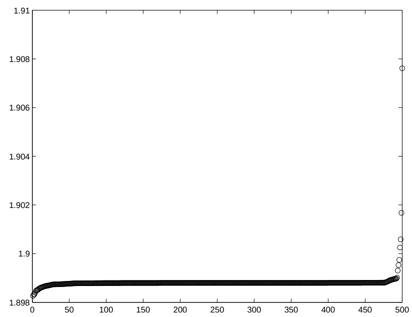

Figure 6 The eigenvalue of the updated mass matrix M

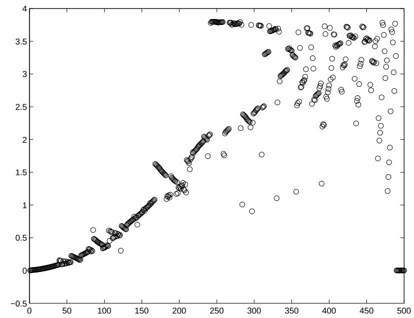

Figure 7 The eigenvalue of the updated stiffness matrix K

After 9 iteration steps, we get the updated mass matrix M, damping matrix C and stiffness matrix K, respectively. The zero/nonzero patterns of M, C and K are plot in Figures 3–5, respectively, which are exactly the same as that of M_a, C_a and K_a. The residual of the dynamics equation is $\|MX\Lambda^2 + CX\Lambda + KX\|_F = 5.0289e-06$ and $\|M - M_a\|_F = 4.4985$, $\|C - C_a\|_F = 3.8717$, $\|K - K_a\|_F = 3.8717$. Finally, by the eigenvalues of M and K, the positive semi-definiteness of M and K are illustrated in Figures 6 and 7, respectively. The numerical results show that our proposed algorithm is efficient for solving the problem (1.4).

4. Conclusion

In this paper, on the assumption that the measured eigenvectors matrix is of full column rank, the mass matrix, damping matrix and stiffness matrix are updated to satisfy the desired properties, including the dynamic equation, symmetric semi-positive definiteness and sparsity requirements. By exploiting the special structure offered by the constraint set, we first reformulate the problem as a constrained optimization problem. Then, the alternating direction method is applied using the separable structure of the problem. The subproblem containing the sparsity constraint is separated into n independent small-scale problems. The results of the numerical examples in structural dynamics show that our proposed algorithm works well.

Acknowledgements The authors would like to express their thanks to the referee for his valuable suggestions and comments.

References

[1] F. TISSEUR, K. MEERBERGEN. *The quadratic eigenvalue problem.* SIAM Rev., 2001, **43**(2): 235–286.

[2] B. N. DATTA. *Numerical Methods for Linear Control Systems.* Elsevier, California, 2003.

[3] M. BARUCH. *Optimization procedure to correct stiffness and flexibility matrices using vibration data.* AIAA. J., 1978, **16**: 1208–1210.

[4] Fushang WEI. *Mass and stiffness interaction effects in analytical model modification.* AIAA. J., 1990, **28**: 1686–1688.

[5] Dongxiu XIE. *A numerical method of structure-perserving model updating problem and its perturbation theory.* Appl. Math. Comput., 2011, **217**: 6364–6371.

[6] M. FRISWELL, J. MOTTERSHEAD. *Finite Element Model Updating in Structural Dynamics.* Kluwer Academic Publishers Group, Dordrecht, 1995.

[7] J. B. CARVALHO, B. N. DATTA, Wenwei LIN, et al. *Symmetry preserving eigenvalue embedding in finite-element model updating of vibrating structures*. J. Sound Vibration, 2006, **290**(3-5): 839–864.

[8] M. FRISWELL, D. INMAN, D. PIKEY. *The direct updating of damping and stiffness matrices*. AIAA. J., 1998, **36**: 491–493.

[9] Y. C. KIO, Wenwei LIN, Shufang XU. *New methods for finite element model updating problems*. AIAA. J., 2006, **44**: 1310–1316.

[10] P. LANCASTER. *Model-updating for self-adjoint quadratic eigenvalue problems*. Linear Algebra Appl., 2008, **428**(11-12): 2778–2790.

[11] Delin CHU, M. CHU, Wenwei LIN. *Quadratic model updating with symmetric, positive definiteness, and no spill-over*. SIAM J Matrx Anal Appl., 2009, **31**: 546–564.

[12] Zhengjian BAI, Meixiang CHEN, Xiaoming YUAN. *Applications of the alternating direction method of multipliers to the semidefinite inverse quadratic eigenvalue problem with partial eigenstructure*. Inverse Problems, 2013, **29**(7): 075011, 27 pp.

[13] Xiantao XIAO, Jian GU, Liwei ZHANG. *Quadratic model updating with gyroscopic structure from partial eigendata*. Optim. Eng., 2013, **14**(3): 431–455.

[14] J. MORENOA, B. N. DATTA, M. RAYDAN. *A symmetry preserving alternating projection method for matrix model updating*. Mechanical Systems and Signal Processing, 2009, **23**(6): 1784–1791.

[15] Kang ZHAO, Lizhi CHENG, Anping LIAO. *Updating $*$-palindromic quadratic systems with no spill-over*. Comput. Appl. Math., 2018, **37**(5): 5587–5608.

[16] Kang ZHAO, Lizhi CHENG, Anping LIAO, et al. *On inverse eigenvalue problems of quadratic palindromic systems with partially prescribed eigenstructure*. Taiwanese J. Math., 2019, **23**(6): 1511–1534.

[17] A. M. KABE. *Stiffness matrix adjustment using mode data*. AIAA. J., 1985, **23**: 1431–1436.

[18] Zhengjian BAI, W. CHING. *A smoothing Newton's method for the construction of a damped vibrating system from noisy test eigendata*. Numer. Linear Algebra Appl., 2009, **16**(2): 109–128.

[19] G. H. GOLUB, C. F. VAN LOAN. *Matrix Computations*. 3rd edition, Johns Hopkins University Press, Baltimore and London, 1996.

[20] N. J. HIGHAM. *Computing a nearest symmetric positive semidefinite matrix*. Linear Algebra Appl., 1988, **103**: 103–118.

[21] Xingju CAI, Guoyong GU, Bingsheng HE, et al. *A proximal point algorithm revisit on the alternating direction method of multipliers*. Sci. China. Math., 2013, **50**(10): 2179–2186.

[22] B. N. DATTA, V. SOKOLOV. *A solution of the affine quadratic inverse eigenvalue problem*. Linear Algebra Appl., 2011, **434**(7): 1745–1760.