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Minimal Critical Sets of Refined Inertias for Irreducible Sign Patterns of Order 3

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Abstract Let S be a nonempty, proper subset of all possible refined inertias of real matrices of order n. The set S is a critical set of refined inertias for irreducible sign patterns of order n, if for each $n \times n$ irreducible sign pattern $\mathcal{A}$, the condition $S \subseteq ri(\mathcal{A})$ is sufficient for $\mathcal{A}$ to be refined inertially arbitrary. If no proper subset of S is a critical set of refined inertias, then S is a minimal critical set of refined inertias for irreducible sign patterns of order n.

All minimal critical sets of refined inertias for full sign patterns of order 3 have been identified in [Wei GAO, Zhongshan LI, Lihua ZHANG, The minimal critical sets of refined inertias for 3×3 full sign patterns, Linear Algebra Appl. 458(2014), 183–196]. In this paper, the minimal critical sets of refined inertias for irreducible sign patterns of order 3 are identified.

Keywords sign pattern; refined inertia; refined inertially arbitrary sign pattern; critical set of refined inertias

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1. Introduction

An $n \times n$ matrix $\mathcal{A}$ is called a sign pattern if its entries are from the set $\{+, -, 0\}$. For a real matrix B, sgn(B) is the sign pattern matrix obtained by replacing each positive (resp., negative, zero) entry of B by + (resp., −, 0). The set of all real matrices with the same sign pattern as the $n \times n$ sign pattern $\mathcal{A}$ is the qualitative class

$$Q(\mathcal{A}) = \{B = [b_{ij}] \in M_n(R) | \text{sgn}(B) = \mathcal{A}\}.$$

A subpattern of an $n \times n$ sign pattern $\mathcal{A}$ is a sign pattern $\mathcal{B}$ obtained by replacing some (possible empty) subset of the nonzero entries of $\mathcal{A}$ with zero. If $\mathcal{B}$ is a subpattern of $\mathcal{A}$, then $\mathcal{A}$ is a superpattern of $\mathcal{B}$.

Let A be a real matrix of order n. The inertia of A is the ordered triple $i(A) = (n_+, n_-, n_0)$, where n_+, n_- and n_0 are the numbers of its eigenvalues (counting multiplicities) with positive, negative and zero real parts, respectively. The refined inertia of A is the ordered quadruple $ri(A) = (n_+, n_-, n_z, 2n_p)$ of nonnegative integers that sum to n, in which $(n_+, n_-, n_z + 2n_p)$ is

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the inertia of A while n_z is the number of 0 as an eigenvalue of A and $2n_p$ is the number of nonzero pure imaginary eigenvalues of A.

For an $n \times n$ sign pattern $\mathcal{A}$, the inertia of $\mathcal{A}$ is $i(\mathcal{A}) = \{i(B)|B \in Q(\mathcal{A})\}$, and the refined inertia of $\mathcal{A}$ is $ri(\mathcal{A}) = \{ri(B)|B \in Q(\mathcal{A})\}$.

The reversal of an inertia (resp., refined inertia) is obtained by exchanging the first two entries in the ordered triple (resp., quadruple), i.e., the reversal of (n_+, n_-, n_0) (resp., $(n_+, n_-, n_z, 2n_p)$) is (n_-, n_+, n_0) (resp., $(n_-, n_+, n_z, 2n_p)$). The reversal of a set of inertias (resp., refined inertias) is the set of reversals of the inertias (resp., refined inertias) in the set. Clearly, for an $n \times n$ sign pattern $\mathcal{A}$, $i(-\mathcal{A})$ is the reversal of $i(\mathcal{A})$ and $ri(-\mathcal{A})$ is the reversal of $ri(\mathcal{A})$.

An $n \times n$ sign pattern $\mathcal{A}$ is called a spectrally arbitrary pattern (SAP) if for each real monic polynomial $r(x)$ of degree n, there exists some $A \in Q(\mathcal{A})$ with characteristic polynomial $p_A(x) = r(x)$. Thus, $\mathcal{A}$ is spectrally arbitrary, if given any self-conjugate spectrum, there exists $A \in Q(\mathcal{A})$ with that spectrum [1].

An $n \times n$ sign pattern $\mathcal{A}$ is called an inertially arbitrary pattern (IAP) if given any ordered triple (n_+, n_-, n_0) of nonnegative integers with $n_+ + n_- + n_0 = n$, there exists a real matrix $A \in Q(\mathcal{A})$ such that $i(A) = (n_+, n_-, n_0)$. Similarly, $\mathcal{A}$ is a refined inertially arbitrary pattern (rIAP) if given any ordered quadruple $(n_+, n_-, n_z, 2n_p)$ of nonnegative integers that sum to n, there exists a real matrix $A \in Q(\mathcal{A})$ such that $ri(A) = (n_+, n_-, n_z, 2n_p)$ (see [2, 3]).

Let S be a nonempty, proper subset of all possible refined inertias of real matrices of order n. Then, S is a critical set of refined inertias for irreducible sign patterns of order n, if for each $n \times n$ irreducible sign pattern $\mathcal{A}$, the condition $S \subseteq ri(\mathcal{A})$ is sufficient for $\mathcal{A}$ to be refined inertially arbitrary.

If no proper subset of S is a critical set of refined inertias for irreducible sign patterns of order n, then S is a minimal critical set of refined inertias for irreducible sign patterns of order n.

A permutation sign pattern is a square sign pattern with entries 0 and +, where the entry + occurs precisely once in each row and in each column. A signature sign pattern is a square diagonal sign pattern each of whose diagonal entries is nonzero. Let $\mathcal{A}$ and $\mathcal{B}$ be two square sign patterns of the same order. We say that $\mathcal{A}$ is permutationally similar to $\mathcal{B}$ if there exists a permutation sign pattern $\mathcal{P}$ such that $\mathcal{B} = \mathcal{P}^T\mathcal{A}\mathcal{P}$, and that $\mathcal{A}$ is signature similar to $\mathcal{B}$ if there exists a signature sign pattern $\mathcal{D}$ such that $\mathcal{B} = \mathcal{D}\mathcal{A}\mathcal{D}$.

Two square sign patterns $\mathcal{A}$ and $\mathcal{B}$ of the same order are equivalent if one can be obtained from the other by any combination of negation, transposition, permutation similarity and signature similarity. Clearly, if $\mathcal{A}$ and $\mathcal{B}$ are equivalent, then $\mathcal{A}$ is an rIAP (resp., IAP) if and only if $\mathcal{B}$ is an rIAP (resp., IAP).

Let $\mathcal{A} = [a_{ij}]$ be an $n \times n$ sign pattern. We say that $\mathcal{A}$ contains a negative 2-cycle (resp., positive 2-cycle) if $a_{ij}a_{ji} = -$ (resp., $a_{ij}a_{ji} = +$) for some $i \neq j$.

Recently, Kim et al. [4] have obtained the minimal critical sets of inertias for irreducible zero-nonzero patterns of order $n = 2, 3, 4$ and for irreducible sign patterns of orders $n = 2, 3$. Yu et al. [5] have given all the minimal critical sets of refined inertias and inertias for irreducible zero-nonzero patterns of order 2 and 3. Also, Yu [6] has identified all the minimal critical sets of refined inertias for irreducible sign patterns of orders 2. All minimal critical sets of refined inertias for full sign patterns of order 3 have been identified [7]. Identifying all minimal critical sets of refined inertias and inertias for irreducible sign patterns that have at least one zero entry has been posed as an open question in [7]. The minimum cardinality of such a set is also open. In this paper, the minimal critical sets of refined inertias and the minimal critical sets of inertias for irreducible sign patterns of order 3 with at least one zero entry are identified.

The main results are the following two theorems.

Theorem 1.1 *The only minimal critical sets of refined inertias for* 3×3 *irreducible sign patterns with at least one zero entry are the following sets and their reversals.*

$$\{(3,0,0,0),(0,3,0,0)\},\{(3,0,0,0),(0,2,1,0)\},\{(3,0,0,0),(0,1,2,0)\},\{(3,0,0,0),(0,1,0,2)\},$$
$$\{(3,0,0,0),(0,0,3,0)\},\{(3,0,0,0),(0,0,1,2)\},\{(2,0,1,0),(0,2,1,0)\},\{(2,0,1,0),(0,1,2,0)\},$$
$$\{(2,0,1,0),(0,1,0,2)\},\{(2,0,1,0),(0,0,3,0)\},\{(2,0,1,0),(0,0,1,2)\},\{(1,0,2,0),(0,1,2,0)\},$$
$$\{(1,0,2,0),(0,1,0,2)\},\{(1,0,2,0),(0,0,3,0)\},\{(1,0,2,0),(0,0,1,2)\},\{(1,0,0,2),(0,1,0,2)\},$$
$$\{(1,0,0,2),(0,0,3,0)\},\{(1,0,0,2),(0,0,1,2)\}.$$

Theorem 1.2 *The only minimal critical sets of inertias for* 3×3 *irreducible sign patterns with at least one zero entry are the following sets and their reversals.*

$$\{(3,0,0),(0,3,0)\},\{(3,0,0),(0,2,1)\},\{(3,0,0),(0,1,2)\},$$
$$\{(3,0,0),(0,0,3)\},\{(2,0,1),(0,2,1)\},\{(2,0,1),(0,1,2)\},$$
$$\{(2,0,1),(0,0,3)\},\{(1,0,2),(0,1,2)\},\{(1,0,2),(0,0,3)\}.$$

The followings are immediate from Theorems 1.1, 1.2 and results of the reference [7].

Theorem 1.3 *The only minimal critical sets of refined inertias for* 3×3 *irreducible sign patterns are the following sets and their reversals.*

$$\{(3,0,0,0),(0,3,0,0)\},\{(3,0,0,0),(0,2,1,0)\},\{(3,0,0,0),(0,1,2,0)\},$$
$$\{(3,0,0,0),(0,1,0,2)\},\{(2,0,1,0),(0,2,1,0)\},\{(2,0,1,0),(0,1,2,0)\},$$
$$\{(2,0,1,0),(0,1,0,2)\},\{(1,0,2,0),(0,1,2,0)\},\{(1,0,0,2),(0,1,0,2)\}.$$

Theorem 1.4 *The only minimal critical sets of inertias for* 3×3 *irreducible sign patterns are the following sets and their reversals.*

$$\{(3,0,0),(0,3,0)\},\{(3,0,0),(0,2,1)\},\{(3,0,0),(0,1,2)\},$$
$$\{(2,0,1),(0,2,1)\},\{(2,0,1),(0,1,2)\},\{(1,0,2),(0,1,2)\}.$$

We will give the proofs of Theorems 1.1 and 1.2 in Sections 3 and 4, respectively.

2. Preliminaries

In this section, we outline some results which are well known for the characterization of 3×3

sign pattern.

Lemma 2.1 ([8]) *If $\mathcal{A}$ is a sign pattern of order 3, then the following statements are equivalent:*

(1) $\mathcal{A}$ is spectrally arbitrary.

(2) $\mathcal{A}$ is inertially arbitrary.

(3) $\mathcal{A}$ is refined inertially arbitrary.

(4) Up to equivalence, $\mathcal{A}$ is a superpattern of one of the following sign pattern:

$$\mathcal{D}_{3,3}=\begin{bmatrix} - & + & 0\\ - & 0 & +\\ - & 0 & + \end{bmatrix},\ \mathcal{D}_{3,2}=\begin{bmatrix} - & + & 0\\ - & 0 & +\\ 0 & - & + \end{bmatrix},\ \mathcal{U}=\begin{bmatrix} - & + & 0\\ - & + & +\\ 0 & + & - \end{bmatrix},\ \mathcal{V}=\begin{bmatrix} - & 0 & +\\ - & 0 & +\\ - & + & + \end{bmatrix}.$$

Lemma 2.2 ([8,9]) *Let*

$$\mathcal{G}=\begin{bmatrix} - & + & +\\ - & + & -\\ - & - & + \end{bmatrix}.$$

Then $\mathcal{G}$ requires a positive eigenvalue.

Lemma 2.3 ([8]) *If φ is a subpattern of $\mathcal{G}$, then φ requires a nonnegative eigenvalue.*

Lemma 2.4 ([3]) *Let m be the maximum number of distinct refined inertias allowed by any sign pattern of order 3. Then $m = 13$.*

3. The minimal critical sets of refined inertias for irreducible sign patterns of order 3

In this section, we identify the minimal critical sets of refined inertias for 3×3 irreducible sign patterns with at least one zero entry.

By Lemma 2.4, there are 13 possible distinct refined inertias for a sign pattern of order 3. We use R to denote the set of these 13 possible distinct refined inertias, that is,

$$R=\{(3,0,0,0),(2,1,0,0),(2,0,1,0),(1,2,0,0),(1,1,1,0),(1,0,2,0),\\ (1,0,0,2),(0,3,0,0),(0,2,1,0),(0,1,2,0),(0,1,0,2),(0,0,3,0),(0,0,1,2)\}.$$

Let $\mathcal{A}$ be an $n \times n$ sign pattern which is not an rIAP. We use $R(\mathcal{A})$ to denote the set of all possible refined inertias that are not in $ri(\mathcal{A})$, that is,

$$R(\mathcal{A}) = R\setminus ri(\mathcal{A}) = \{(n_+, n_-, n_z, 2n_p) \in Z_+^4 | n_+ + n_- + n_z + 2n_p = n, (n_+, n_-, n_z, 2n_p) \notin ri(\mathcal{A})\},$$

where Z_+ is the set of all nonnegative integers.

For convenience, write

$$R_0 = \{(0,0,3,0),(0,0,1,2)\},$$

$$R_1 = \{(3,0,0,0),(2,0,1,0),(1,0,2,0),(1,0,0,2)\},$$

$$R_1' = \{(0,3,0,0),(0,2,1,0),(0,1,2,0),(0,1,0,2)\},$$

where R_1' is the reversal of R_1. Let

$$\mathcal{P}_{13}=\begin{bmatrix}0&0&+\\0&+&0\\+&0&0\end{bmatrix},\mathcal{P}_{23}=\begin{bmatrix}+&0&0\\0&0&+\\0&+&0\end{bmatrix},\mathcal{P}_{12}=\begin{bmatrix}0&+&0\\+&0&0\\0&0&+\end{bmatrix},$$

and $\mathcal{D}_1=\mathrm{diag}(-,+,+)$, $\mathcal{D}_2=\mathrm{diag}(+,-,+)$, $\mathcal{D}_3=\mathrm{diag}(+,+,-)$.

Lemma 3.1 *Let $\mathcal{A}$ be a 3×3 irreducible sign pattern with at least one zero entry. Suppose all diagonal entries of $\mathcal{A}$ are nonzero, and the diagonal entries of $\mathcal{A}$ have different signs. If $\mathcal{A}$ is not an rIAP, then one of the following conditions holds.*

(1) $R_0\cup R_1\subseteq R(\mathcal{A})$;

(2) $R_0\cup R_1'\subseteq R(\mathcal{A})$.

Proof Up to equivalence, we can assume that $a_{11}=-$, $a_{22}=+$ and $a_{33}=+$. Note that $\mathcal{A}$ is irreducible (this means $\mathcal{A}$ has at most three zero entries), and has at least one zero entry. We consider the following three cases.

Case 1. Exactly one off-diagonal entry of $\mathcal{A}$ is zero.

Up to equivalence, $\mathcal{A}$ has the following forms.

$$(1.1)\begin{bmatrix}-&*&*\\ *&+&*\\0&*&+\end{bmatrix},\quad (1.2)\begin{bmatrix}-&*&*\\ *&+&*\\ *&0&+\end{bmatrix},\quad (1.1)'\begin{bmatrix}-&*&*\\0&+&*\\ *&*&+\end{bmatrix},$$

where $*\in\{+,-\}$. Noting that $\mathcal{P}_{23}(1.1)'\mathcal{P}_{23}=(1.1)$, so $(1.1)'$ and (1.1) are equivalent.

Let $\mathcal{A}=[a_{ij}]$ have form (1.1). If $a_{12}<0$, taking $\mathcal{A}'=\mathcal{D}_2\mathcal{A}\mathcal{D}_2$, then $\mathcal{A}'$ and $\mathcal{A}$ are equivalent and the $(1,2)$ entry of $\mathcal{A}'$ is positive. If $a_{13}<0$, taking $\mathcal{A}''=\mathcal{D}_3\mathcal{A}\mathcal{D}_3$, then $\mathcal{A}''$ and $\mathcal{A}$ are equivalent and the $(1,3)$ entry of $\mathcal{A}''$ is positive. So, without loss of generality, we can take $a_{12}=a_{13}=+$.

According to the number of negative 2-cycles, $\mathcal{A}$ is possibly one of the following sign patterns.

$$\mathcal{A}_1=\begin{bmatrix}-&+&+\\+&+&+\\0&+&+\end{bmatrix},\mathcal{A}_2=\begin{bmatrix}-&+&+\\+&+&-\\0&-&+\end{bmatrix},\mathcal{A}_3=\begin{bmatrix}-&+&+\\-&+&+\\0&+&+\end{bmatrix},\mathcal{A}_4=\begin{bmatrix}-&+&+\\-&+&-\\0&-&+\end{bmatrix},$$

$$\mathcal{A}_5=\begin{bmatrix}-&+&+\\+&+&+\\0&-&+\end{bmatrix},\mathcal{A}_6=\begin{bmatrix}-&+&+\\+&+&-\\0&+&+\end{bmatrix},\mathcal{A}_7=\begin{bmatrix}-&+&+\\-&+&+\\0&-&+\end{bmatrix},\mathcal{A}_8=\begin{bmatrix}-&+&+\\-&+&-\\0&+&+\end{bmatrix}.$$

Let $\mathcal{A}=[a_{ij}]$ have form (1.2). Without loss of generality, we can take $a_{12}=a_{13}=+$. According to the number of negative 2-cycles, $\mathcal{A}$ is possibly one of the following sign patterns.

$$\mathcal{A}_9=\begin{bmatrix}-&+&+\\+&+&+\\+&0&+\end{bmatrix},\mathcal{A}_{10}=\begin{bmatrix}-&+&+\\+&+&-\\+&0&+\end{bmatrix},\mathcal{A}_{11}=\begin{bmatrix}-&+&+\\-&+&+\\+&0&+\end{bmatrix},\mathcal{A}_{12}=\begin{bmatrix}-&+&+\\-&+&-\\+&0&+\end{bmatrix},$$

$$\mathcal{A}_{13}=\begin{bmatrix} - & + & + \\ + & + & + \\ - & 0 & + \end{bmatrix}, \mathcal{A}_{14}=\begin{bmatrix} - & + & + \\ + & + & - \\ - & 0 & + \end{bmatrix}, \mathcal{A}_{15}=\begin{bmatrix} - & + & + \\ - & + & + \\ - & 0 & + \end{bmatrix}, \mathcal{A}_{16}=\begin{bmatrix} - & + & + \\ - & + & - \\ - & 0 & + \end{bmatrix}.$$

Firstly, let us notice the following facts.

(1) $\mathcal{A}_{15}$, $\mathcal{D}_2\mathcal{P}_{13}(-\mathcal{A}_6)\mathcal{P}_{13}\mathcal{D}_2$, $\mathcal{D}_3\mathcal{A}_{12}\mathcal{D}_3$, $\mathcal{D}_1\mathcal{P}_{23}\mathcal{A}_{13}^T\mathcal{P}_{23}\mathcal{D}_1$ are superpatterns of $\mathcal{D}_{3,3}$.

(2) $\mathcal{A}_7$, $\mathcal{D}_3\mathcal{A}_8\mathcal{D}_3$ are superpatterns of $\mathcal{D}_{3,2}$.

(3) $\mathcal{D}_1\mathcal{P}_{23}\mathcal{A}_3^T\mathcal{P}_{23}\mathcal{D}_1$, $-\mathcal{D}_3\mathcal{P}_{13}\mathcal{P}_{23}\mathcal{A}_{11}^T\mathcal{P}_{23}\mathcal{P}_{13}\mathcal{D}_3$, $\mathcal{D}_2\mathcal{P}_{13}(-\mathcal{A}_{14})\mathcal{P}_{13}\mathcal{D}_2$ are superpatterns of $\mathcal{V}$.

Then by Lemma 2.1, $\mathcal{A}_3$, $\mathcal{A}_6$, $\mathcal{A}_7$, $\mathcal{A}_8$, $\mathcal{A}_{11}$, $\mathcal{A}_{12}$, $\mathcal{A}_{13}$, $\mathcal{A}_{14}$ and $\mathcal{A}_{15}$ are rIAPs.

Thus, $\mathcal{A}$ is equivalent to one of $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_4$, $\mathcal{A}_5$, $\mathcal{A}_9$, $\mathcal{A}_{10}$ and $\mathcal{A}_{16}$.

Now, we consider $\mathcal{A}_1$. For any $A \in Q(\mathcal{A}_1)$, we may assume A has been scaled so that $a_{33}=1$. We may also assume that $a_{12}=a_{23}=1$ (otherwise they can be 1 by suitable similarities). Thus, assume

$$A=\begin{bmatrix} -a & 1 & b \\ c & d & 1 \\ 0 & e & 1 \end{bmatrix} \in Q(\mathcal{A}_1),$$

where $a,b,c,d,e>0$. Then the characteristic polynomial of A is

$$p_A(x)=x^3+(a-d-1)x^2+(-ad-a-c+d-e)x+ad-ae+c-bce.$$

By the relationship between the coefficients of the characteristic polynomial and eigenvalues, if $ri(A)\in R_0\cup R_1'=\{(0,3,0,0),(0,2,1,0),(0,1,2,0),(0,1,0,2),(0,0,3,0),(0,0,1,2)\}$, then

$$\begin{cases} a-d-1\geq 0, \\ -ad-a-c+d-e\geq 0. \end{cases}$$

Adding both sides of above two inequalities, respectively, we have $-ad-c-e-1\geq 0$. It is a contradiction. Hence $R_0\cup R_1'\subseteq R(\mathcal{A}_1)$.

By similar argument to $\mathcal{A}_1$, we can get $R_0\cup R_1'\subseteq R(\mathcal{A}_i)$ for $i=2,9,10$.

For $\mathcal{A}_4$, without loss of generality, let

$$A=\begin{bmatrix} -a & 1 & b \\ -c & d & -1 \\ 0 & -e & 1 \end{bmatrix} \in Q(\mathcal{A}_4),$$

where $a,b,c,d,e>0$. Then the characteristic polynomial of A is

$$p_A(x)=x^3+(a-d-1)x^2+(-ad-a+c+d-e)x+ad-ae-c-bce.$$

If $ri(A)\in R_0\cup R_1'$, then

$$\begin{cases} a-d-1\geq 0, \\ -ad-a+c+d-e\geq 0, \\ ad-ae-c-bce\geq 0. \end{cases}$$

Adding both sides of above three inequalities, respectively, we have $-1-e-ae-bce\geq 0$. It is a contradiction. Hence, $R_0\cup R_1'\subseteq R(\mathcal{A}_4)$.

Noting that $\mathcal{A}_5$ requires negative determinant, we get $R_0\cup R_1\subseteq R(\mathcal{A}_5)$.

For $\mathcal{A}_{16}$, noting that $\mathcal{A}_{16}$ is a subpattern of $\mathcal{G}$, by Lemma 2.3, we know that $(0,3,0,0)$ and $(0,1,0,2)$ are not the refined inertias of $\mathcal{A}_{16}$. In the following, we show that $(0,2,1,0)$, $(0,1,2,0)$, $(0,0,3,0)$ and $(0,0,1,2)$ are not the refined inertias of $\mathcal{A}_{16}$.

For any $A \in Q(\mathcal{A}_{16})$, without loss of generality, let

$$A = \begin{bmatrix} -a & 1 & b \\ -c & d & -1 \\ -e & 0 & 1 \end{bmatrix},$$

where $a, b, c, d, e > 0$. Then the characteristic polynomial of A is $p_A(x) = x^3 + p_1x^2 + p_2x + p_3$, where

$$\begin{cases} p_1 = -1 + a - d, \\ p_2 = -a + c + d - ad + be, \\ p_3 = -c + ad - e - bde. \end{cases}$$

If $ri(A) \in \{(0,2,1,0),(0,1,2,0),(0,0,3,0),(0,0,1,2)\}$, then $p_3 = 0$, $p_1 \geq 0$ and $p_2 \geq 0$. By $p_3 = 0$, we have $a = \frac{c+e+bde}{d}$. Then

$$\begin{cases} dp_1 = -d + c + e + bde - d^2 \geq 0, \\ dp_2 = -(c + e + bde) + cd + d^2 - (c + e + bde)d + bed \\ \qquad = -c - e + d^2 - ed - bed^2 \geq 0. \end{cases}$$

If $be \geq 1$, then $dp_2 = -c - e + d^2 - ed - bed^2 \leq -c - e - ed < 0$.

If $c + e > d^2$, then $dp_2 = -c - e + d^2 - ed - bed^2 < -ed - bed^2 < 0$.

If $be < 1$ and $c + e \leq d^2$, then $dp_1 = -d + c + e + bde - d^2 < 0$.

All of above are contradictions. Hence, $(0,2,1,0)$, $(0,1,2,0)$, $(0,0,3,0)$ and $(0,0,1,2)$ are not the refined inertias of $\mathcal{A}_{16}$ and so $R_0 \cup R_1' \subseteq R(\mathcal{A}_{16})$.

Case 2. Exactly two off-diagonal entries of $\mathcal{A}$ are zero.

According to whether the two zero entries are in one 2-cycle or not, up to equivalence, $\mathcal{A}$ has the following forms

$$(2.1) \begin{bmatrix} - & * & 0 \\ * & + & * \\ 0 & * & + \end{bmatrix}, (2.2) \begin{bmatrix} - & * & * \\ * & + & 0 \\ * & 0 & + \end{bmatrix}, (2.1)' \begin{bmatrix} - & 0 & * \\ 0 & + & * \\ * & * & + \end{bmatrix},$$

$$(2.3) \begin{bmatrix} - & 0 & * \\ * & + & * \\ 0 & * & + \end{bmatrix}, (2.4) \begin{bmatrix} - & * & * \\ * & + & 0 \\ 0 & * & + \end{bmatrix}, (2.4)' \begin{bmatrix} - & 0 & * \\ * & + & 0 \\ * & * & + \end{bmatrix},$$

where $* \in \{+,-\}$. Noting that $\mathcal{P}_{23}(2.1)'\mathcal{P}_{23} = (2.1)$, $\mathcal{P}_{23}((2.4)')^T\mathcal{P}_{23} = (2.4)$, so $(2.1)'$ and (2.1), $(2.4)'$ and (2.4) are equivalent, respectively.

Let $\mathcal{A} = [a_{ij}]$ have form (2.1). If $a_{12} < 0$, taking $\mathcal{A}' = \mathcal{D}_1\mathcal{A}\mathcal{D}_1$, then $\mathcal{A}'$ and $\mathcal{A}$ are equivalent and the $(1,2)$ entry of $\mathcal{A}'$ is positive. If $a_{23} < 0$, taking $\mathcal{A}'' = \mathcal{D}_3\mathcal{A}\mathcal{D}_3$, then $\mathcal{A}''$ and $\mathcal{A}$ are equivalent and the $(2,3)$ entry of $\mathcal{A}''$ is positive. So, without loss of generality, we can take $a_{12} = a_{23} = +$.

According to the number of the negative 2-cycles, $\mathcal{A}$ is possibly one of the following sign patterns.

$$\mathcal{A}_{17} = \begin{bmatrix} - & + & 0 \\ + & + & + \\ 0 & + & + \end{bmatrix}, \mathcal{A}_{18} = \begin{bmatrix} - & + & 0 \\ - & + & + \\ 0 & + & + \end{bmatrix}, \mathcal{A}_{19} = \begin{bmatrix} - & + & 0 \\ + & + & + \\ 0 & - & + \end{bmatrix}, \mathcal{A}_{20} = \begin{bmatrix} - & + & 0 \\ - & + & + \\ 0 & - & + \end{bmatrix}.$$

Let $\mathcal{A}$ have form (2.2). Without loss of generality, we can let $a_{12} = a_{13} = +$. According to the number of the negative 2-cycles, $\mathcal{A}$ is possibly one of the following sign patterns.

$$\mathcal{A}_{21} = \begin{bmatrix} - & + & + \\ + & + & 0 \\ + & 0 & + \end{bmatrix}, \mathcal{A}_{22} = \begin{bmatrix} - & + & + \\ - & + & 0 \\ + & 0 & + \end{bmatrix}, \mathcal{A}_{23} = \begin{bmatrix} - & + & + \\ + & + & 0 \\ - & 0 & + \end{bmatrix}, \mathcal{A}_{24} = \begin{bmatrix} - & + & + \\ - & + & 0 \\ - & 0 & + \end{bmatrix}.$$

Let $\mathcal{A} = [a_{ij}]$ have form (2.3). If $a_{13} < 0$, taking $\mathcal{A}' = \mathcal{D}_1\mathcal{A}\mathcal{D}_1$, then $\mathcal{A}'$ and $\mathcal{A}$ are equivalent and the $(1,3)$ entry of $\mathcal{A}'$ is positive. If $a_{23} < 0$, taking $\mathcal{A}'' = \mathcal{D}_2\mathcal{A}\mathcal{D}_2$, then $\mathcal{A}''$ and $\mathcal{A}$ are equivalent and the $(2,3)$ entry of $\mathcal{A}''$ is positive. So, without loss of generality, we can take $a_{13} = a_{23} = +$.

According to the number of the negative 2-cycles, $\mathcal{A}$ is possibly one of the following sign patterns.

$$\mathcal{A}_{25} = \begin{bmatrix} - & 0 & + \\ + & + & + \\ 0 & + & + \end{bmatrix}, \mathcal{A}_{26} = \begin{bmatrix} - & 0 & + \\ - & + & + \\ 0 & + & + \end{bmatrix}, \mathcal{A}_{27} = \begin{bmatrix} - & 0 & + \\ + & + & + \\ 0 & - & + \end{bmatrix}, \mathcal{A}_{28} = \begin{bmatrix} - & 0 & + \\ - & + & + \\ 0 & - & + \end{bmatrix}.$$

Assume that $\mathcal{A}$ has form (2.4). Without loss of generality, we can let $a_{12} = a_{13} = +$. According to the number of the negative 2-cycles, $\mathcal{A}$ is possibly one of the following sign patterns.

$$\mathcal{A}_{29} = \begin{bmatrix} - & + & + \\ + & + & 0 \\ 0 & + & + \end{bmatrix}, \mathcal{A}_{30} = \begin{bmatrix} - & + & + \\ + & + & 0 \\ 0 & - & + \end{bmatrix}, \mathcal{A}_{31} = \begin{bmatrix} - & + & + \\ - & + & 0 \\ 0 & + & + \end{bmatrix}, \mathcal{A}_{32} = \begin{bmatrix} - & + & + \\ - & + & 0 \\ 0 & - & + \end{bmatrix}.$$

Firstly, let us notice the following facts.

(1) $\mathcal{P}_{13}(-\mathcal{A}_{28})\mathcal{P}_{13}$, $\mathcal{D}_1\mathcal{A}_{31}^T\mathcal{D}_1$ are superpatterns of $\mathcal{D}_{3,3}$.

(2) $\mathcal{A}_{20}$ is a superpattern of $\mathcal{D}_{3,2}$.

(3) $\mathcal{D}_3\mathcal{P}_{12}(-\mathcal{A}_{22})\mathcal{P}_{12}\mathcal{D}_3 = \mathcal{U}$.

(4) $\mathcal{P}_{23}\mathcal{A}_{23}\mathcal{P}_{23} = \mathcal{A}_{22}$.

Then by Lemma 2.1, $\mathcal{A}_{20}$, $\mathcal{A}_{22}$, $\mathcal{A}_{23}$, $\mathcal{A}_{28}$ and $\mathcal{A}_{31}$ are rIAPs.

Thus, $\mathcal{A}$ is equivalent to one of patterns in Case 2 except for $\mathcal{A}_{20}$, $\mathcal{A}_{22}$, $\mathcal{A}_{23}$, $\mathcal{A}_{28}$ and $\mathcal{A}_{31}$.

By similar argument to $\mathcal{A}_1$ in Case 1, we can get $R_0 \cup R_1' \subseteq R(\mathcal{A}_i)$ for $i = 17, 21, 25, 26, 29, 30$.

By similar argument to $\mathcal{A}_4$ in Case 1, we can get $R_0 \cup R_1' \subseteq R(\mathcal{A}_i)$ for $i = 18, 32$.

Noting that $\mathcal{A}_{19}$ and $\mathcal{A}_{27}$ require negative determinants, we get $R_0 \cup R_1 \subseteq R(\mathcal{A}_i)$ for $i = 19, 27$.

For $\mathcal{A}_{24}$, noting that it is a subpattern of $\mathcal{G}$, by Lemma 2.3, we know that $(0,3,0,0)$ and $(0,1,0,2)$ do not belong to the refined inertias of $\mathcal{A}_{24}$. In the following, we prove that $(0,2,1,0)$, $(0,1,2,0)$, $(0,0,3,0)$ and $(0,0,1,2)$ do not belong to the refined inertias of $\mathcal{A}_{24}$ as well.

For any $A \in Q(\mathcal{A}_{24})$, without loss of generality, let

$$A = \begin{bmatrix} -a & 1 & b \\ -c & d & 0 \\ -e & 0 & 1 \end{bmatrix},$$

where $a, b, c, d, e > 0$. Then the characteristic polynomial of A is $p_A(x) = x^3 + p_1x^2 + p_2x + p_3$, where

$$\begin{cases} p_1 = -1 + a - d, \\ p_2 = -a + c + d - ad + be, \\ p_3 = -c + ad - bde. \end{cases}$$

If $ri(A) \in \{(0, 2, 1, 0), (0, 1, 2, 0), (0, 0, 3, 0), (0, 0, 1, 2)\}$, then $p_3 = 0$, $p_1 \geq 0$ and $p_2 \geq 0$. By $p_3 = 0$, we have $a = \frac{c+bde}{d}$. Then

$$\begin{cases} dp_1 = -d + c + bde - d^2 \geq 0, \\ dp_2 = -(c + bde) + cd + d^2 - (c + bde)d + bed = -c + d^2 - bed^2 \geq 0. \end{cases}$$

If $be \geq 1$, then $dp_2 = -c + d^2 - bed^2 \leq -c < 0$.

If $c > d^2$, then $dp_2 = -c + d^2 - bed^2 < -bed^2 < 0$.

If $be < 1$ and $c \leq d^2$, then $dp_1 = -d + c + bde - d^2 < 0$.

All of above are contradictions. Hence, $(0, 2, 1, 0)$, $(0, 1, 2, 0)$, $(0, 0, 3, 0)$ and $(0, 0, 1, 2)$ do not belong to the refined inertias of $\mathcal{A}_{24}$. Thus $R_0 \cup R_1' \subseteq R(\mathcal{A}_{24})$.

Case 3. Exactly three off-diagonal entries of $\mathcal{A}$ are zero.

Up to equivalence, $\mathcal{A}$ has the following unique form

$$(3.1) \begin{bmatrix} - & 0 & * \\ * & + & 0 \\ 0 & * & + \end{bmatrix},$$

where $* \in \{+, -\}$.

Let $\mathcal{A} = [a_{ij}]$ have form (3.1). If $a_{13} < 0$, taking $\mathcal{A}' = \mathcal{D}_1\mathcal{A}\mathcal{D}_1$, then $\mathcal{A}'$ and $\mathcal{A}$ are equivalent and the $(1, 3)$ entry of $\mathcal{A}'$ is positive. If $a_{32} < 0$, taking $\mathcal{A}'' = \mathcal{D}_2\mathcal{A}\mathcal{D}_2$, then $\mathcal{A}''$ and $\mathcal{A}$ are equivalent and the $(3, 2)$ entry of $\mathcal{A}''$ is positive. So, without loss of generality, we can take $a_{13} = a_{32} = +$.

Then $\mathcal{A}$ is possibly one of the following sign patterns.

$$\mathcal{A}_{33} = \begin{bmatrix} - & 0 & + \\ + & + & 0 \\ 0 & + & + \end{bmatrix}, \quad \mathcal{A}_{34} = \begin{bmatrix} - & 0 & + \\ - & + & 0 \\ 0 & + & + \end{bmatrix}.$$

By similar argument to $\mathcal{A}_1$ in Case 1, we can get $R_0 \cup R_1' \subseteq R(\mathcal{A}_{33})$.

Noting that $\mathcal{A}_{34}$ requires negative determinant, we get $R_0 \cup R_1 \subseteq R(\mathcal{A}_{34})$. □

Lemma 3.2 *Let $\mathcal{A}$ be a 3×3 irreducible sign pattern with at least one zero entry. Suppose $\mathcal{A}$ has one zero diagonal entry, and two nonzero diagonal entries have different signs. If $\mathcal{A}$ is not an rIAP, then one of the following conditions holds.*

(1) $R_0 \cup R_1 \subseteq R(\mathcal{A})$*;*

(2) $R_0 \cup R_1' \subseteq R(\mathcal{A})$*.*

Proof Up to equivalence, we can assume $a_{11} = -$, $a_{22} = 0$ and $a_{33} = +$. Note that $\mathcal{A}$ is irreducible and has at least one zero entry. We consider the following three cases.

Case 1. All off-diagonal entries of $\mathcal{A}$ are nonzero.

Without loss of generality, we can take $a_{12} = a_{13} = +$. According to the number of negative 2-cycles, $\mathcal{A}$ is possibly one of the following sign patterns.

$$\mathcal{A}_1 = \begin{bmatrix} - & + & + \\ + & 0 & + \\ + & + & + \end{bmatrix}, \mathcal{A}_2 = \begin{bmatrix} - & + & + \\ + & 0 & - \\ + & - & + \end{bmatrix}, \mathcal{A}_3 = \begin{bmatrix} - & + & + \\ - & 0 & + \\ + & + & + \end{bmatrix}, \mathcal{A}_4 = \begin{bmatrix} - & + & + \\ - & 0 & - \\ + & - & + \end{bmatrix},$$

$$\mathcal{A}_5 = \begin{bmatrix} - & + & + \\ + & 0 & + \\ - & + & + \end{bmatrix}, \mathcal{A}_6 = \begin{bmatrix} - & + & + \\ + & 0 & - \\ - & - & + \end{bmatrix}, \mathcal{A}_7 = \begin{bmatrix} - & + & + \\ + & 0 & + \\ + & - & + \end{bmatrix}, \mathcal{A}_8 = \begin{bmatrix} - & + & + \\ + & 0 & - \\ + & + & + \end{bmatrix},$$

$$\mathcal{A}_9 = \begin{bmatrix} - & + & + \\ - & 0 & + \\ - & + & + \end{bmatrix}, \mathcal{A}_{10} = \begin{bmatrix} - & + & + \\ - & 0 & - \\ - & - & + \end{bmatrix}, \mathcal{A}_{11} = \begin{bmatrix} - & + & + \\ - & 0 & + \\ + & - & + \end{bmatrix}, \mathcal{A}_{12} = \begin{bmatrix} - & + & + \\ - & 0 & - \\ + & + & + \end{bmatrix},$$

$$\mathcal{A}_{13} = \begin{bmatrix} - & + & + \\ + & 0 & + \\ - & - & + \end{bmatrix}, \mathcal{A}_{14} = \begin{bmatrix} - & + & + \\ + & 0 & - \\ - & + & + \end{bmatrix}, \mathcal{A}_{15} = \begin{bmatrix} - & + & + \\ - & 0 & + \\ - & - & + \end{bmatrix}, \mathcal{A}_{16} = \begin{bmatrix} - & + & + \\ - & 0 & - \\ - & + & + \end{bmatrix}.$$

Firstly, let us notice the following facts.

(1) $\mathcal{A}_{11}$, $\mathcal{A}_{15}$, $\mathcal{D}_3\mathcal{A}_{12}\mathcal{D}_3$, $\mathcal{D}_3\mathcal{A}_{16}\mathcal{D}_3$ are superpatterns of $\mathcal{D}_{3,2}$.

(2) $\mathcal{D}_3\mathcal{A}_4\mathcal{D}_3$, $\mathcal{D}_3\mathcal{D}_2\mathcal{A}_3^T\mathcal{D}_2\mathcal{D}_3$ are superpatterns of $\mathcal{D}_{3,3}$.

(3) $\mathcal{A}_9$, $\mathcal{D}_1\mathcal{A}_5^T\mathcal{D}_1$, $\mathcal{D}_2\mathcal{A}_6\mathcal{D}_2$, $\mathcal{D}_2\mathcal{P}_{13}(-\mathcal{A}_{14})\mathcal{P}_{13}\mathcal{D}_2$ are superpatterns of $\mathcal{V}$.

(4) $\mathcal{A}_8^T = \mathcal{A}_7$, $\mathcal{D}_3\mathcal{P}_{13}(-\mathcal{A}_7)\mathcal{P}_{13}\mathcal{D}_3 = \mathcal{A}_3$.

Then by Lemma 2.1, $\mathcal{A}_3$, $\mathcal{A}_4$, $\mathcal{A}_5$, $\mathcal{A}_6$, $\mathcal{A}_7$, $\mathcal{A}_8$, $\mathcal{A}_9$, $\mathcal{A}_{11}$, $\mathcal{A}_{12}$, $\mathcal{A}_{14}$, $\mathcal{A}_{15}$ and $\mathcal{A}_{16}$ are rIAPs.

Thus, in this case, $\mathcal{A}$ is equivalent to one of $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_{10}$ and $\mathcal{A}_{13}$.

For $\mathcal{A}_1$, without loss of generality, let

$$A = \begin{bmatrix} -a & 1 & b \\ c & 0 & 1 \\ d & e & 1 \end{bmatrix} \in Q(\mathcal{A}_1),$$

where $a, b, c, d, e > 0$. Then the characteristic polynomial of A is

$$p_A(x) = x^3 + (a-1)x^2 + (-c - bd - e - a)x + c - d - ae - bce.$$

Since $-c - bd - e - a < 0$, we have $n_+(A) \geq 1$ and $n_-(A) \geq 1$. Then $R_0 \cup R_1' \subseteq R(\mathcal{A}_1)$.

By similar argument to $\mathcal{A}_1$, we can get $R_0 \cup R_1' \subseteq R(\mathcal{A}_2)$.

Noting that $\mathcal{A}_{10}$ requires positive determinant, we get $R_0 \cup R_1' \subseteq R(\mathcal{A}_{10})$.

Noting that $\mathcal{A}_{13}$ requires negative determinant, we get $R_0 \cup R_1 \subseteq R(\mathcal{A}_{13})$.

Case 2. Exactly one off-diagonal entry of $\mathcal{A}$ is zero.

Up to equivalence, $\mathcal{A}$ has the following forms

$$(2.1)\begin{bmatrix} - & * & * \\ * & 0 & * \\ 0 & * & + \end{bmatrix},\quad (2.2)\begin{bmatrix} - & * & * \\ * & 0 & * \\ * & 0 & + \end{bmatrix},\quad (2.2)'\begin{bmatrix} - & * & * \\ 0 & 0 & * \\ * & * & + \end{bmatrix},$$

where $* \in \{+,-\}$. Noting that $\mathcal{P}_{13}(-(2.2)')^T\mathcal{P}_{13} = (2.2)$, so $(2.2)'$ and (2.2) are equivalent.

Let $\mathcal{A}$ have form (2.1). Without loss of generality, let $a_{12} = a_{23} = +$. According to the number of the negative 2-cycles, $\mathcal{A}$ is possibly one of the following sign patterns.

$$\mathcal{A}_{17} = \begin{bmatrix} - & + & + \\ + & 0 & + \\ 0 & + & + \end{bmatrix}, \mathcal{A}_{18} = \begin{bmatrix} - & + & - \\ + & 0 & + \\ 0 & + & + \end{bmatrix}, \mathcal{A}_{19} = \begin{bmatrix} - & + & + \\ - & 0 & + \\ 0 & + & + \end{bmatrix}, \mathcal{A}_{20} = \begin{bmatrix} - & + & - \\ - & 0 & + \\ 0 & + & + \end{bmatrix},$$

$$\mathcal{A}_{21} = \begin{bmatrix} - & + & + \\ + & 0 & + \\ 0 & - & + \end{bmatrix}, \mathcal{A}_{22} = \begin{bmatrix} - & + & - \\ + & 0 & + \\ 0 & - & + \end{bmatrix}, \mathcal{A}_{23} = \begin{bmatrix} - & + & + \\ - & 0 & + \\ 0 & - & + \end{bmatrix}, \mathcal{A}_{24} = \begin{bmatrix} - & + & - \\ - & 0 & + \\ 0 & - & + \end{bmatrix}.$$

Let $\mathcal{A}$ have form (2.2). Without loss of generality, let $a_{12} = a_{13} = +$. According to the number of the negative 2-cycles, $\mathcal{A}$ is possibly one of the following sign patterns.

$$\mathcal{A}_{25} = \begin{bmatrix} - & + & + \\ + & 0 & + \\ + & 0 & + \end{bmatrix}, \mathcal{A}_{26} = \begin{bmatrix} - & + & + \\ + & 0 & - \\ + & 0 & + \end{bmatrix}, \mathcal{A}_{27} = \begin{bmatrix} - & + & + \\ - & 0 & + \\ + & 0 & + \end{bmatrix}, \mathcal{A}_{28} = \begin{bmatrix} - & + & + \\ - & 0 & - \\ + & 0 & + \end{bmatrix},$$

$$\mathcal{A}_{29} = \begin{bmatrix} - & + & + \\ + & 0 & + \\ - & 0 & + \end{bmatrix}, \mathcal{A}_{30} = \begin{bmatrix} - & + & + \\ + & 0 & - \\ - & 0 & + \end{bmatrix}, \mathcal{A}_{31} = \begin{bmatrix} - & + & + \\ - & 0 & + \\ - & 0 & + \end{bmatrix}, \mathcal{A}_{32} = \begin{bmatrix} - & + & + \\ - & 0 & - \\ - & 0 & + \end{bmatrix}.$$

Firstly, let us notice the following facts.

(1) $\mathcal{A}_{31}$, $\mathcal{D}_1\mathcal{A}_{19}^T\mathcal{D}_1$, $\mathcal{D}_3\mathcal{P}_{13}(-\mathcal{A}_{22})\mathcal{P}_{13}\mathcal{D}_3$, $\mathcal{D}_3\mathcal{A}_{28}\mathcal{D}_3$ are the superpatterns of $\mathcal{D}_{3,3}$.

(2) $\mathcal{A}_{23}$, $\mathcal{A}_{24}$ are the superpatterns of $\mathcal{D}_{3,2}$.

(3) $\mathcal{P}_{13}\mathcal{D}_2(-\mathcal{A}_{30})\mathcal{D}_2\mathcal{P}_{13} = \mathcal{V}$.

Then by Lemma 2.1, $\mathcal{A}_{19}$, $\mathcal{A}_{22}$, $\mathcal{A}_{23}$, $\mathcal{A}_{24}$, $\mathcal{A}_{28}$, $\mathcal{A}_{30}$ and $\mathcal{A}_{31}$ are rIAPs.

Thus, in this case, $\mathcal{A}$ is equivalent to one of patterns in Case 2 except for $\mathcal{A}_{19}$, $\mathcal{A}_{22}$, $\mathcal{A}_{23}$, $\mathcal{A}_{24}$, $\mathcal{A}_{28}$, $\mathcal{A}_{30}$ and $\mathcal{A}_{31}$.

By similar argument to $\mathcal{A}_1$ in Case 1, we can get $R_0 \cup R_1' \subseteq R(\mathcal{A}_i)$ for $i = 17, 18, 25$.

Noting that $\mathcal{A}_{20}$, $\mathcal{A}_{27}$ and $\mathcal{A}_{32}$ require positive determinants, we get $R_0 \cup R_1' \subseteq R(\mathcal{A}_i)$ for $i = 20, 27, 32$.

Noting that $\mathcal{A}_{21}$, $\mathcal{A}_{26}$ and $\mathcal{A}_{29}$ require negative determinants, we get $R_0 \cup R_1 \subseteq R(\mathcal{A}_i)$ for $i = 21, 26, 29$.

Case 3. Exactly two off-diagonal entries of $\mathcal{A}$ are zero.

According to whether the two zero entries are in one 2-cycle or not, up to equivalence, $\mathcal{A}$ has the following forms

$$(3.1)\begin{bmatrix} - & * & 0 \\ * & 0 & * \\ 0 & * & + \end{bmatrix}, (3.2)\begin{bmatrix} - & * & * \\ * & 0 & 0 \\ * & 0 & + \end{bmatrix}, (3.2)'\begin{bmatrix} - & 0 & * \\ 0 & 0 & * \\ * & * & + \end{bmatrix},$$

$$(3.3)\begin{bmatrix} - & 0 & * \\ * & 0 & * \\ 0 & * & + \end{bmatrix}, (3.3)'\begin{bmatrix} - & * & * \\ * & 0 & 0 \\ 0 & * & + \end{bmatrix}, (3.4)\begin{bmatrix} - & * & * \\ 0 & 0 & * \\ * & 0 & + \end{bmatrix},$$

where $* \in \{+,-\}$. Noting that $\mathcal{P}_{13}(-(3.2)')\mathcal{P}_{13} = (3.2)$, $\mathcal{P}_{13}(-(3.3)')^T\mathcal{P}_{13} = (3.3)$, so $(3.2)'$ and (3.2), $(3.3)'$ and (3.3) are equivalent, respectively.

Let $\mathcal{A}$ have form (3.1). Without loss of generality, let $a_{12} = a_{23} = +$. According to the number of the negative 2-cycles, $\mathcal{A}$ is possibly one of the following sign patterns.

$$\mathcal{A}_{33} = \begin{bmatrix} - & + & 0 \\ + & 0 & + \\ 0 & + & + \end{bmatrix}, \mathcal{A}_{34} = \begin{bmatrix} - & + & 0 \\ + & 0 & + \\ 0 & - & + \end{bmatrix}, \mathcal{A}_{35} = \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & + & + \end{bmatrix}, \mathcal{A}_{36} = \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & - & + \end{bmatrix}.$$

Let $\mathcal{A}$ have form (3.2). Without loss of generality, let $a_{12} = a_{13} = +$. According to the number of the negative 2-cycles, $\mathcal{A}$ is possibly one of the following sign patterns.

$$\mathcal{A}_{37} = \begin{bmatrix} - & + & + \\ + & 0 & 0 \\ + & 0 & + \end{bmatrix}, \mathcal{A}_{38} = \begin{bmatrix} - & + & + \\ - & 0 & 0 \\ + & 0 & + \end{bmatrix}, \mathcal{A}_{39} = \begin{bmatrix} - & + & + \\ + & 0 & 0 \\ - & 0 & + \end{bmatrix}, \mathcal{A}_{40} = \begin{bmatrix} - & + & + \\ - & 0 & 0 \\ - & 0 & + \end{bmatrix}.$$

Let $\mathcal{A}$ have form (3.3). Without loss of generality, let $a_{13} = a_{23} = +$. According to the number of the negative 2-cycles, $\mathcal{A}$ is possibly one of the following sign patterns.

$$\mathcal{A}_{41} = \begin{bmatrix} - & 0 & + \\ + & 0 & + \\ 0 & + & + \end{bmatrix}, \mathcal{A}_{42} = \begin{bmatrix} - & 0 & + \\ - & 0 & + \\ 0 & + & + \end{bmatrix}, \mathcal{A}_{43} = \begin{bmatrix} - & 0 & + \\ + & 0 & + \\ 0 & - & + \end{bmatrix}, \mathcal{A}_{44} = \begin{bmatrix} - & 0 & + \\ - & 0 & + \\ 0 & - & + \end{bmatrix}.$$

Let $\mathcal{A}$ have form (3.4). Without loss of generality, let $a_{12} = a_{13} = +$. According to the number of the negative 2-cycles, $\mathcal{A}$ is possibly one of the following sign patterns.

$$\mathcal{A}_{45} = \begin{bmatrix} - & + & + \\ 0 & 0 & + \\ + & 0 & + \end{bmatrix}, \mathcal{A}_{46} = \begin{bmatrix} - & + & + \\ 0 & 0 & - \\ + & 0 & + \end{bmatrix}, \mathcal{A}_{47} = \begin{bmatrix} - & + & + \\ 0 & 0 & + \\ - & 0 & + \end{bmatrix}, \mathcal{A}_{48} = \begin{bmatrix} - & + & + \\ 0 & 0 & - \\ - & 0 & + \end{bmatrix}.$$

Firstly, let us notice the following facts.

(1) $\mathcal{P}_{13}(-\mathcal{A}_{44})\mathcal{P}_{13}$ is a superpattern of $\mathcal{D}_{3,3}$.

(2) $\mathcal{A}_{36}$ is $\mathcal{D}_{3,2}$.

Then by Lemma 2.1, $\mathcal{A}_{36}$ and $\mathcal{A}_{44}$ are rIAPs.

Thus in this case, $\mathcal{A}$ is equivalent to one pattern in Case 3 except for $\mathcal{A}_{36}$ and $\mathcal{A}_{44}$.

By similar argument to $\mathcal{A}_1$ in Case 1, we can get $R_0 \cup R_1' \subseteq R(\mathcal{A}_i)$ for $i = 33, 42$.

Noting that $\mathcal{A}_{34}$, $\mathcal{A}_{37}$, $\mathcal{A}_{39}$, $\mathcal{A}_{43}$, $\mathcal{A}_{46}$ and $\mathcal{A}_{47}$ require negative determinants, we get $R_0 \cup R_1 \subseteq R(\mathcal{A}_i)$ for $i = 34, 37, 39, 43, 46, 47$.

Noting that $\mathcal{A}_{35}$, $\mathcal{A}_{38}$, $\mathcal{A}_{40}$, $\mathcal{A}_{41}$, $\mathcal{A}_{45}$ and $\mathcal{A}_{48}$ require positive determinants, we get $R_0 \cup R_1' \subseteq R(\mathcal{A}_i)$ for $i = 35, 38, 40, 41, 45, 48$.

Case 4. Exactly three off-diagonal entries of $\mathcal{A}$ are zero.

Up to equivalence, $\mathcal{A}$ has the following unique form

$$\begin{bmatrix} - & 0 & * \\ * & 0 & 0 \\ 0 & * & + \end{bmatrix}, \tag{4.1}$$

where $* \in \{+, -\}$.

It is easy to see that $\mathcal{A}$ is sign nonsingular. If $\mathcal{A}$ requires positive determinant, then $R_0 \cup R_1' \subseteq R(\mathcal{A})$. If $\mathcal{A}$ requires negative determinant, then $R_0 \cup R_1 \subseteq R(\mathcal{A})$. $\square$

Theorem 3.3 *Let $\mathcal{A}$ be a 3×3 irreducible sign pattern with at least one zero entry. If $\mathcal{A}$ is not an rIAP, then one of the following conditions holds:*

(1) $R_0 \cup R_1 \subseteq R(\mathcal{A})$;

(2) $R_0 \cup R_1' \subseteq R(\mathcal{A})$;

(3) $R_1 \cup R_1' \subseteq R(\mathcal{A})$.

Proof Let $\mathcal{A}$ be a 3×3 irreducible sign pattern with at least one zero entry. Suppose $\mathcal{A}$ is not an rIAP. We consider the following cases.

Case 1. All diagonal entries of $\mathcal{A}$ are zero.

Since $\mathcal{A}$ requires the zero trace, $(3, 0, 0, 0)$, $(2, 0, 1, 0)$, $(0, 1, 2, 0)$, $(0, 1, 0, 2)$, $(0, 3, 0, 0)$, $(0, 2, 1, 0)$, $(0, 1, 2, 0)$, and $(0, 1, 0, 2)$ do not belong to the refined inertias of $\mathcal{A}$, and so $R_1 \cup R_1' \subseteq R(\mathcal{A})$.

Case 2. Sign pattern $\mathcal{A}$ has at least one nonzero diagonal entry, and all nonzero diagonal entries of $\mathcal{A}$ have the same sign.

If all nonzero diagonal entries of $\mathcal{A}$ are negative, then $\mathcal{A}$ requires the negative trace, and so $R_0 \cup R_1 \subseteq R(\mathcal{A})$. If all nonzero diagonal entries of $\mathcal{A}$ are positive, then $\mathcal{A}$ requires the positive trace, and so $R_0 \cup R_1' \subseteq R(\mathcal{A})$.

Case 3. Sign pattern $\mathcal{A}$ has at least two nonzero diagonal entries, and the nonzero diagonal entries of $\mathcal{A}$ have different signs.

By Lemmas 3.1 and 3.2, we know the result holds. $\square$

Theorem 3.4 *There exists a 3×3 irreducible sign pattern $\mathcal{A}$ with at least one zero entry such that $\mathcal{A}$ is not an rIAP, and $R(\mathcal{A}) = R_1 \cup R_1'$.*

Proof Let

$$\mathcal{S}_1 = \begin{bmatrix} 0 & + & + \\ - & 0 & + \\ + & + & 0 \end{bmatrix}.$$

It is easy to see that $\mathcal{S}_1$ is not an rIAP. Since $\mathcal{S}_1$ requires the zero trace, so $R_1 \cup R_1' \subseteq R(\mathcal{S}_1)$.

On the other hand, for any $A \in Q(\mathcal{S}_1)$, we may assume that $a_{12} = a_{13} = 1$ (otherwise they can be 1 by suitable similarities). Thus, without loss of generality, assume

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -a & 0 & b \\ c & d & 0 \end{bmatrix},$$

where $a, b, c, d > 0$.

By taking suitable values of a, b, c, d shown in Table 1, we can find real matrices in $Q(\mathcal{S}_1)$ with each refined inertia in $R \setminus (R_1 \cup R_1')$.

refined inertia	a	b	c	d
(2, 1, 0, 0)	2	1	1	1
(1, 2, 0, 0)	1	2	1	1
(1, 1, 1, 0)	1	1	1	1
(0, 0, 3, 0)	1	1	$\frac{1}{2}$	$\frac{1}{2}$
(0, 0, 1, 2)	4	2	1	$\frac{1}{2}$

Table 1 Realization of each refined inertia in $R \setminus (R_1 \cup R_1')$

Theorem 3.5 *There exists a 3×3 irreducible sign pattern $\mathcal{A}$ with at least one zero entry such that $\mathcal{A}$ is not an rIAP, and $R(\mathcal{A}) = R_0 \cup R_1'$.*

Proof Let

$$\mathcal{S}_2 = \begin{bmatrix} - & + & + \\ - & + & - \\ 0 & - & + \end{bmatrix}.$$

Noting that $\mathcal{S}_2$ is the sign pattern $\mathcal{A}_4$ in the proof of Lemma 3.1, by Lemma 3.1, $\mathcal{S}_2$ is not an rIAP and $R_0 \cup R_1' \subseteq R(\mathcal{S}_2)$.

On the other hand, take

$$A = \begin{bmatrix} -a & 1 & b \\ -c & d & -1 \\ 0 & -e & 1 \end{bmatrix} \in Q(\mathcal{S}_2),$$

where $a, b, c, d, e > 0$. By taking suitable values of a, b, c, d, e shown in Table 2, we can find real matrices in $Q(\mathcal{S}_2)$ with each refined inertia in $R \setminus (R_0 \cup R_1')$. □

Theorem 3.6 *There exists a 3×3 irreducible sign pattern $\mathcal{A}$ with at least one zero entry such that $\mathcal{A}$ is not an rIAP, and $R(\mathcal{A}) = R_0 \cup R_1$.*

Proof Let $\mathcal{S}_3 = -\mathcal{S}_2$. By Theorem 3.5, the result follows. □

Lemma 3.7 ([7]) *Let H be a proper subset of set of all possible refined inertias of real matrices of order n. Then H is a critical set of refined inertias for a family $\mathcal{F}$ of sign pattern of order n if and only if every $n \times n$ sign pattern $\mathcal{A}$ in $\mathcal{F}$ that is not an rIAP, $H \cap R(\mathcal{A}) \neq \emptyset$.*

refined inertia	a	b	c	d	e
$(3,0,0,0)$	1	1	5	3	1
$(2,0,1,0)$	1	1	2	$\frac{7}{2}$	$\frac{1}{2}$
$(1,0,2,0)$	2	$\frac{1}{5}$	20	16	2
$(1,0,0,2)$	1	1	2	$\frac{7}{3}$	$\frac{1}{2}$
$(2,1,0,0)$	1	1	2	11	$\frac{1}{2}$
$(1,2,0,0)$	1	1	1	1	1
$(1,1,1,0)$	3	2	1	2	1

Table 2 Realization of each refined inertia in $R \setminus (R_0 \cup R_1')$

Theorem 3.8 *Let H be a proper subset of R. Then H is a critical set of refined inertias for irreducible sign patterns of order 3 with at least one zero entry if and only if one of the following conditions holds:*

(1) $H \cap R_1 \neq \emptyset$ and $H \cap R_1' \neq \emptyset$;

(2) $H \cap R_0 \neq \emptyset$ and $H \cap R_1 \neq \emptyset$;

(3) $H \cap R_0 \neq \emptyset$ and $H \cap R_1' \neq \emptyset$.

Proof Let $\mathcal{F}$ be the set of all irreducible sign patterns of order 3 with at least one zero entry that are not rIAPs. By Lemma 3.7, we only need to prove $H \cap R(\mathcal{A}) \neq \emptyset$ for every $\mathcal{A}$ in $\mathcal{F}$ if and only if one of the following conditions holds:

(1) $H \cap R_1 \neq \emptyset$ and $H \cap R_1' \neq \emptyset$;

(2) $H \cap R_0 \neq \emptyset$ and $H \cap R_1 \neq \emptyset$;

(3) $H \cap R_0 \neq \emptyset$ and $H \cap R_1' \neq \emptyset$.

By Theorem 3.3, the sufficiency is clear.

For the necessity, let $H \cap R(\mathcal{A}) \neq \emptyset$ for every $\mathcal{A}$ in $\mathcal{F}$. Then by Theorems 3.4–3.6, $H \cap (R_1 \cup R_1') \neq \emptyset$, $H \cap (R_0 \cup R_1) \neq \emptyset$, and $H \cap (R_0 \cup R_1') \neq \emptyset$. So the necessity holds. □

Proof of Theorem 1.1 Let H be a proper subsets of the set of all possible refined inertias for irreducible sign patterns of order 3 with at least one zero entry. By Theorem 3.8, H is critical set of refined inertias if and only if $H \cap R_1 \neq \emptyset$ and $H \cap R_1' \neq \emptyset$, or $H \cap R_0 \neq \emptyset$ and $H \cap R_1 \neq \emptyset$, or $H \cap R_0 \neq \emptyset$ and $H \cap R_1' \neq \emptyset$.

To make H a minimal critical set, then one of the following conditions holds:

(1) $|H \cap R_1| = 1$ and $|H \cap R_1'| = 1$;

(2) $|H \cap R_0| = 1$ and $|H \cap R_1| = 1$;

(3) $|H \cap R_0| = 1$ and $|H \cap R_1'| = 1$.

We pick up exactly one refined inertia from R_1 and one refined inertia from $R_0 \cup R_1'$, or one refined inertia from R_1' and one refined inertia from R_0, and let them form new sets as follows.

$$\{(3,0,0,0),(0,3,0,0)\}, \{(3,0,0,0),(0,2,1,0)\}, \{(3,0,0,0),(0,1,2,0)\}, \{(3,0,0,0),(0,1,0,2)\},$$
$$\{(3,0,0,0),(0,0,3,0)\}, \{(3,0,0,0),(0,0,1,2)\}, \{(2,0,1,0),(0,3,0,0)\}, \{(2,0,1,0),(0,2,1,0)\},$$
$$\{(2,0,1,0),(0,1,2,0)\}, \{(2,0,1,0),(0,1,0,2)\}, \{(2,0,1,0),(0,0,3,0)\}, \{(2,0,1,0),(0,0,1,2)\},$$

$\{(1,0,2,0),(0,3,0,0)\},\{(1,0,2,0),(0,2,1,0)\},\{(1,0,2,0),(0,1,2,0)\},\{(1,0,2,0),(0,1,0,2)\},$
$\{(1,0,2,0),(0,0,3,0)\},\{(1,0,2,0),(0,0,1,2)\},\{(1,0,0,2),(0,3,0,0)\},\{(1,0,0,2),(0,2,1,0)\},$
$\{(1,0,0,2),(0,1,2,0)\},\{(1,0,0,2),(0,1,0,2)\},\{(1,0,0,2),(0,0,3,0)\},\{(1,0,0,2),(0,0,1,2)\},$
$\{(0,3,0,0),(0,0,3,0)\},\{(0,3,0,0),(0,0,1,2)\},\{(0,2,1,0),(0,0,3,0)\},\{(0,2,1,0),(0,0,1,2)\},$
$\{(0,1,2,0),(0,0,3,0)\},\{(0,1,2,0),(0,0,1,2)\},\{(0,1,0,2),(0,0,3,0)\},\{(0,1,0,2),(0,0,1,2)\}.$

Note that
$\{(2,0,1,0),(0,3,0,0)\}$ is the reversal of $\{(3,0,0,0),(0,2,1,0)\}$,
$\{(1,0,2,0),(0,3,0,0)\}$ is the reversal of $\{(3,0,0,0),(0,1,2,0)\}$,
$\{(1,0,2,0),(0,2,1,0)\}$ is the reversal of $\{(2,0,1,0),(0,1,2,0)\}$,
$\{(1,0,0,2),(0,3,0,0)\}$ is the reversal of $\{(3,0,0,0),(0,1,0,2)\}$,
$\{(1,0,0,2),(0,2,1,0)\}$ is the reversal of $\{(2,0,1,0),(0,1,0,2)\}$,
$\{(1,0,0,2),(0,1,2,0)\}$ is the reversal of $\{(1,0,2,0),(0,1,0,2)\}$,
$\{(0,3,0,0),(0,0,3,0)\}$ is the reversal of $\{(3,0,0,0),(0,0,3,0)\}$,
$\{(0,3,0,0),(0,0,1,2)\}$ is the reversal of $\{(3,0,0,0),(0,0,1,2)\}$,
$\{(0,2,1,0),(0,0,3,0)\}$ is the reversal of $\{(2,0,1,0),(0,0,3,0)\}$,
$\{(0,2,1,0),(0,0,1,2)\}$ is the reversal of $\{(2,0,1,0),(0,0,1,2)\}$,
$\{(0,1,2,0),(0,0,3,0)\}$ is the reversal of $\{(1,0,2,0),(0,0,3,0)\}$,
$\{(0,1,2,0),(0,0,1,2)\}$ is the reversal of $\{(1,0,2,0),(0,0,1,2)\}$,
$\{(0,1,0,2),(0,0,3,0)\}$ is the reversal of $\{(1,0,0,2),(0,0,3,0)\}$,
$\{(0,1,0,2),(0,0,1,2)\}$ is the reversal of $\{(1,0,0,2),(0,0,1,2)\}$.
So we drop them out.

Theorem 1.1 now follows. □

By Theorem 1.1, it is clear that the maximum cardinality of a minimum critical set of refined inertias for 3×3 irreducible sign patterns with at least one zero entry is 2.

4. The minimal critical sets of inertias for irreducible sign patterns of order 3

Using the same method as in the proof of Theorem 1.2 in [7], we can get the result of Theorem 1.2.

5. Summary and conclusions

In this paper, we obtained all the minimum critical sets of refined inertias and inertias for 3×3 irreducible sign patterns with at least one zero entry. Based on these conclusions and results from the reference [7], we identified all the minimum critical sets of refined inertias and inertias for 3×3 irreducible sign patterns.

Further topics of interest for future research include the investigation of all the minimal critical sets of refined inertias and inertias for sign patterns of order n ($n \geq 4$) and other parameters for sign patterns (see for instance the recent results in [10]).

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